

# Determination of time-dependent coefficients for a hyperbolic inverse problem

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## Abstract

We consider an inverse boundary value problem for the hyperbolic partial differential equation

$$(-i\partial_t + A_0(t, x))^2 u(t, x) - \sum_{j=1}^n (-i\partial_{x_j} + A_j(t, x))^2 u(t, x) + V(t, x)u(t, x) = 0$$

with time dependent vector and scalar potentials ( $\mathcal{A} = (A_0, \dots, A_m)$  and  $V(t, x)$  respectively) on a bounded, smooth cylindric domain  $(-\infty, \infty) \times \Omega$ . Using a geometric optics construction we show that the boundary data allows us to recover integrals of the potentials along ‘light rays’ and we then establish the uniqueness of these potentials modulo a gauge transform. Also, a logarithmic stability estimate is obtained and the presence of obstacles inside the domain is studied. In this case, it is shown that under some geometric restrictions similar uniqueness results hold.

## 1 Introduction

Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$ ,  $n \geq 2$ , consider the hyperbolic equation with time dependent coefficients

$$(-i\partial_t + A_0(t, x))^2 u - \sum_{j=1}^n (-i\partial_{x_j} + A_j(t, x))^2 u + V(t, x)u = 0 \quad \text{in } \mathbb{R} \times \Omega, \quad (1)$$

where  $V(t, x)$ ,  $A_j(t, x)$ ,  $0 \leq j \leq n$ , are smooth functions vanishing when  $\{|x| > R\}$  for some  $R > 0$ . The smooth vector field  $\mathcal{A}(t, x) = (A_0(t, x), \dots, A_n(t, x))$  is called the *vector potential*, the function  $V(t, x)$  is called the *scalar potential* and equation (1) is often referred to as the *relativistic Schrödinger equation* (see [24]).

For the above differential equation we impose the initial and boundary conditions

$$u(t, x) = \partial_t u(t, x) = 0 \quad \text{for } t \ll 0 \quad (2)$$

$$u(t, x) = f(t, x) \quad \text{on } \mathbb{R} \times \partial\Omega, \quad (3)$$

where  $f$  is a compactly supported smooth function on  $\mathbb{R} \times \partial\Omega$ . Solutions to (1) satisfying (2) and (3) are unique and we can define the *Dirichlet to Neumann* operator by

$$\Lambda(f) := (\partial_\nu + iA(t, x) \cdot \nu) u(t, x) \Big|_{\mathbb{R} \times \partial\Omega} \quad (4)$$

where  $u$  is the solution of (1)-(3),  $\nu$  is the exterior unit normal to  $\partial\Omega$  and we have set  $A(t, x) = (A_1(t, x), \dots, A_n(t, x))$ . The *Inverse Boundary Value Problem* is the recovery of  $\mathcal{A}(t, x)$  and  $V(t, x)$  knowing  $\Lambda(f)$  for all  $f \in C_0^\infty(\mathbb{R} \times \partial\Omega)$ .

Inverse problems is a topic in mathematics that has been growing in interest for the past decades, in part, due to its wide range of applications, from medicine to acoustics to electromagnetism just to mention a few (see for instance [14] for some of the latest tools and techniques employed in the solutions of these problems). In the case of the hyperbolic inverse boundary value problem (1)-(4) with time independent coefficients, a powerful tool called the *boundary control method*, or BC-method for short, was discovered by Belishev (see [3]). It was later developed by Belishev, Kurylev, Lassas, and others ([17],[18]), and more recently a new approach to this problem based on the BC-method was developed by Eskin in ([5],[6]). On a similar note, Stefanov and Uhlmann established uniqueness and stability results for the wave equation in anisotropic media (see [26] and [30] for a survey of these results).

Nevertheless, the case of time dependent coefficients has seen very little progress in recent years. In the case of the vector potential being identically equal to zero ( $\mathcal{A} \equiv 0$  in (1)), Stefanov [25] and Ramm-Sjöstrand [23], have shown that the Dirichlet to Neumann map completely determines the scalar

potentials. More recently, Eskin [7] considered the case of time-dependent potentials that are analytic in time. The analyticity of the time variable is related to the use of a unique continuation theorem established by Tataru in [28]. In this paper we eliminate the restriction on the analyticity in the time variable and not only we extend the uniqueness results in [25], [23], but we also establish a logarithmic stability estimate for the case when the vector potentials are compactly supported in space and time.

We also study the problem with obstacles inside the domain and show that under some geometric considerations similar uniqueness results hold. The presence of these obstacles in the domain may lead to the Aharonov-Bohm effect. This problem was considered by Nicoleau and Weder in the context of the inverse scattering (see [20] and [31] respectively), and by Eskin [8] in the context of the inverse boundary value problem for the Schrödinger equation.

This work is structured as follows. In section 2 we introduce the notion of *gauge equivalent* for a pair of vector and scalar potentials and we make some remarks about uniqueness. In section 3 we construct geometric optic solutions (GO for short) for equation (1) satisfying the set of initial conditions (2). In section 4 we establish a Green's formula for these types of problems and show that the light ray transforms of gauge equivalent potentials agree. In section 6 we prove uniqueness of the potentials in the case where no obstacles are allowed inside the domain  $\Omega$ . In section 7, based on the works of Isakov [13], Isakov and Sun [15] and more recently Begmatov [2], we establish a stability result of logarithmic type for the particular case when the potentials are compactly supported in both space and time. Finally in section 8 we consider the problem when one or more convex bodies are allowed inside the domain (by imposing a geometric restriction on the layout of these obstacles).

## 2 Gauge equivalence

The ultimate goal in most inverse boundary value problems is the recovery of the coefficients of a partial differential equation, however for application purposes this recovery is meaningless unless it can be done in some sort of 'unique' way. In our case this type of uniqueness is obtained modulo a *gauge transform*.

**Definition 2.1.** We say that the vector and scalar potentials  $(\mathcal{A}(t, x), V(t, x))$  and  $(\mathcal{A}'(t, x), V'(t, x))$  are *gauge equivalent* if there exists  $g(t, x) \in C^\infty(\mathbb{R} \times \overline{\Omega})$  such that  $g(t, x) \neq 0$  on  $\mathbb{R} \times \overline{\Omega}$ ,  $g = 1$  on  $\mathbb{R} \times \partial\Omega$  and

$$\begin{aligned}\mathcal{A}'(t, x) &= \mathcal{A}(t, x) - \frac{i}{g(t, x)} \nabla_{t,x} g(t, x) \\ V'(t, x) &= V(t, x),\end{aligned}$$

where  $\nabla_{t,x} := (\partial_t, \partial_x) = (\partial_t, \partial_{x_1}, \dots, \partial_{x_n})$  is the  $(n+1)$ -dimensional gradient. The mapping  $(\mathcal{A}, V) \rightarrow (\mathcal{A}', V')$  is called a *gauge transform*.

The definition above includes the more general case when obstacles are present inside the domain. When  $\Omega$  is simply connected (no obstacles), the gauge  $g$  has the particular form  $g(t, x) = e^{i\varphi(t, x)}$  where  $\varphi(t, x) \in C^\infty(\mathbb{R} \times \Omega)$ . Then  $-\frac{i}{g(t, x)} \nabla_{(t,x)} g(t, x) = \nabla_{(t,x)} \varphi(t, x)$  and we see that two vector potentials are gauge equivalent if their difference is the gradient of a smooth function. The following proposition tells us that recovery of the potentials can only be done up to a gauge transform.

**Proposition 2.1.** *If  $u(t, x)$  is a solution of (1)-(3) and  $g(t, x)$  is as in definition (2.1), then  $v(t, x) = g(t, x)u(t, x)$  satisfies*

$$\begin{aligned}(-i\partial_t + A'_0(t, x))^2 v - \sum_{j=1}^n (-i\partial_{x_j} + A'_j(t, x))^2 v + V'(t, x)v &= 0 \text{ in } \mathbb{R} \times \Omega \\ v &= \partial_t v = 0 \text{ for } t \leq T_1 \\ v &= fg|_{\mathbb{R} \times \partial\Omega} \text{ on } \mathbb{R} \times \partial\Omega.\end{aligned}\tag{5}$$

with  $(\mathcal{A}', V')$  and  $(\mathcal{A}, V)$  gauge equivalent.

In addition if  $\Lambda'$  is the Dirichlet to Neumann operator associated to (5), then

$$\Lambda'(v|_{\mathbb{R} \times \partial\Omega}) = g|_{\mathbb{R} \times \partial\Omega} \Lambda(u|_{\mathbb{R} \times \partial\Omega})\tag{6}$$

i.e.,  $\Lambda' = \Lambda$  since  $g|_{\mathbb{R} \times \partial\Omega} = 1$ .

*Proof.* Setting  $x_0 = t$  we see that for  $0 \leq j \leq n$ ,

$$(-i\partial_{x_j} + A'_j(t, x)) v(t, x) = g(t, x) \left( -i\partial_{x_j} + A'_j(t, x) - \frac{i}{g(t, x)} \partial_{x_j} g(t, x) \right) u(t, x).$$

Choosing  $A'_j(t, x)$  so that  $A'_j = A_j + \frac{i}{g} \partial_{x_j} g$  for  $0 \leq j \leq n$ , we get

$$\begin{aligned} (-i\partial_{x_j} + A'_j(t, x))^2 v(t, x) &= (-i\partial_{x_j} + A'_j(t, x)) (g(t, x) (-i\partial_{x_j} + A_j(t, x)) u(t, x)) \\ &= g(t, x) (-i\partial_{x_j} + A_j(t, x))^2 u(t, x), \end{aligned}$$

thus

$$\begin{aligned} &(-i\partial_t + A'_0(t, x))^2 v - \sum_{j=1}^n (-i\partial_{x_j} + A'_j(t, x))^2 v + V'(t, x)v = \\ &g(t, x) \left( (-i\partial_t + A_0(t, x))^2 u - \sum_{j=1}^n (-i\partial_{x_j} + A_j(t, x))^2 u + V(t, x)u \right) = 0 \end{aligned}$$

as  $u$  is a solution of (1). Also notice that since  $g$  is smooth and  $u$  satisfies (2) and (3) we have for  $t < 0$

$$\begin{aligned} v(t, x) &= u(t, x)g(t, x) = 0 \\ \partial_t v(t, x) &= u(t, x)\partial_t g(t, x) + \partial_t u(t, x)g(t, x) = 0, \end{aligned}$$

similarly  $v(t, x)|_{\mathbb{R} \times \partial\Omega} = (g(t, x)u(t, x))|_{\mathbb{R} \times \partial\Omega} = fg|_{\mathbb{R} \times \partial\Omega}$ . To conclude we simply notice that

$$\begin{aligned} \Lambda' \left( v|_{\mathbb{R} \times \partial\Omega} \right) &= (\partial_\nu(gu) + iA' \cdot \nu(gu))|_{\mathbb{R} \times \partial\Omega} \\ &= \left( (\partial_\nu g)u + g(\partial_\nu u) + i(A + ig^{-1}\partial_x g) \cdot \nu(gu) \right)|_{\mathbb{R} \times \partial\Omega} \\ &= g(\partial_\nu u + i(A \cdot \nu)u)|_{\mathbb{R} \times \partial\Omega} + ((\partial_\nu g)u - (\partial_\nu g)u)|_{\mathbb{R} \times \partial\Omega} \\ &= g|_{\mathbb{R} \times \partial\Omega} \Lambda \left( u|_{\mathbb{R} \times \partial\Omega} \right) \end{aligned}$$

□

If the above equality holds we shall say that the Dirichlet to Neumann maps  $\Lambda$  and  $\Lambda'$  are gauge equivalent. Summarizing, we have shown that if the vector and scalar potentials are gauge equivalent then the Dirichlet to Neumann maps are equal. In the following pages we shall attempt to prove the converse, roughly speaking: *If for a pair of vector and scalar potentials the Dirichlet to Neumann operators associated to the hyperbolic equation (1)-(3) are equal, then so are the vector and scalar potentials.*

### 3 Geometric optics

For the hyperbolic problem (1)-(3) we shall attempt to construct geometric optics solutions supported near light rays. In order for us to achieve this goal we consider solutions having the form

$$u(t, x) = e^{ik(t-\omega \cdot x)} \sum_{p=0}^N \frac{v_p(t, x)}{(2ik)^p} + v^{(N+1)}(t, x), \quad \omega \in S^{n-1}, k \in \mathbb{R}. \quad (7)$$

For  $u$  as above we have

$$(-i\partial_t + A_0)u = e^{ik(t-\omega \cdot x)} (-i\partial_t + A_0 + ik) \left( \sum_{p=0}^N \frac{v_p}{(2ik)^p} + e^{-ik(t-\omega \cdot x)} v^{(N+1)} \right) \quad (8)$$

applying the above identity twice to a solution of (1) we get

$$\begin{aligned} (-i\partial_t + A_0)^2 u &= (-i\partial_t + A_0) \left( e^{ik(t-\omega \cdot x)} (-i\partial_t + A_0 + ik) \left( \sum_{p=0}^N \frac{v_p}{(2ik)^p} + \right. \right. \\ &\quad \left. \left. e^{-ik(t-\omega \cdot x)} v^{(N+1)} \right) \right) \\ &= e^{ik(t-\omega \cdot x)} (-i\partial_t + A_0 + ik) (-i\partial_t + A_0 + ik) \left( \sum_{p=0}^N \frac{v_p}{(2ik)^p} + \right. \\ &\quad \left. e^{-ik(t-\omega \cdot x)} v^{(N+1)} \right) \end{aligned}$$

this is

$$\begin{aligned} (-i\partial_t + A_0)^2 u &= e^{ik(t-\omega \cdot x)} \left( (-i\partial_t + A_0)^2 + \right. \\ &\quad \left. 2ik(-i\partial_t + A_0) - k^2 \right) \left( \sum_{p=0}^N \frac{v_p}{(2ik)^p} + e^{-ik(t-\omega \cdot x)} v^{(N+1)} \right). \end{aligned}$$

Since a similar formula holds for  $1 \leq j \leq n$ , we obtain

$$\begin{aligned} (-i\partial_{x_j} + A_j)^2 u &= e^{ik(t-\omega \cdot x)} \left( (-i\partial_{x_j} + A_j)^2 + \right. \\ &\quad \left. 2ik\omega_j(-i\partial_{x_j} + A_j) - k^2\omega_j^2 \right) \left( \sum_{p=0}^N \frac{v_p}{(2ik)^p} + e^{-ik(t-\omega \cdot x)} v^{(N+1)} \right), \end{aligned}$$

thus equation (1) becomes

$$\begin{aligned}
0 = Lu &= e^{ik(t-\omega \cdot x)} \left( (-i\partial_t + A_0)^2 - \sum_{j=1}^n (-i\partial_{x_j} + A_j)^2 + V \right) v \\
&\quad + 2ik e^{ik(t-\omega \cdot x)} \left( (-i\partial_t + A_0) + \sum_{j=1}^n \omega_j (-i\partial_{x_j} + A_j) \right) v \\
&\quad + e^{ik(t-\omega \cdot x)} \left( -k^2 + \sum_{j=1}^n (\omega_j k)^2 \right) v \\
0 = Lu &= e^{ik(t-\omega \cdot x)} (L + 2ik\mathcal{L})v,
\end{aligned} \tag{9}$$

where we have set

$$v(t, x) = \sum_{p=0}^N \frac{v_p(t, x)}{(2ik)^p} + e^{-ik(t-\omega \cdot x)} v^{(N+1)}(t, x) \tag{10}$$

$$L = (-i\partial_t + A_0(t, x))^2 - \sum_{j=1}^n (-i\partial_{x_j} + A_j(t, x))^2 + V(t, x) \tag{11}$$

$$\mathcal{L} = (-i\partial_t + A_0(t, x)) + \sum_{j=1}^n \omega_j (-i\partial_{x_j} + A_j(t, x)). \tag{12}$$

Plugging in the expression for  $v$  into (9) we obtain

$$0 = (2ik\mathcal{L} + L) \left( v_0 + \frac{1}{(2ik)}v_1 + \cdots + \frac{1}{(2ik)^N}v_N + e^{-ik(t-\omega \cdot x)}v^{(N+1)} \right), \tag{13}$$

which in turn can be rewritten as

$$\begin{aligned}
(2ik)\mathcal{L}v_0 &+ (\mathcal{L}v_1 + Lv_0) + \frac{1}{(2ik)}(\mathcal{L}v_2 + Lv_1) + \cdots + \\
&\frac{1}{(2ik)^{N-1}}(\mathcal{L}v_N + Lv_{N-1}) + \frac{1}{(2ik)^N}Lv_N + e^{-ik(t-\omega \cdot x)}Lv^{(N+1)} = 0,
\end{aligned} \tag{14}$$

where we have used the identity  $(2ik\mathcal{L} + L)(e^{-ik(t-\omega \cdot x)}v^{(N+1)}) = e^{-ik(t-\omega \cdot x)}Lv^{(N+1)}$ .

Notice that a solution of (1)-(2) can be found by solving the  $N + 1$  transport equations

$$\mathcal{L}v_0 = 0, \quad \mathcal{L}v_j = -Lv_{j-1}, \quad 1 \leq j \leq N \tag{15}$$

with initial conditions supported near a neighborhood of the light ray  $\gamma = \{(t', x') + s(1, \omega) : (t', x') \perp (1, \omega), s \in \mathbb{R}\}$  (we assume that  $\gamma$  intersects the plane  $t = T_1$  outside of the cylinder  $\mathbb{R} \times \Omega$ ; as well as the second order equation

$$Lv^{(N+1)} = -\frac{e^{ik(t-\omega \cdot x)}}{(2ik)^{N+1}}Lv_N \quad (16)$$

with initial and boundary conditions

$$\begin{aligned} v^{(N+1)}(t, x) &= 0 && \text{for } t = T_1 \\ \partial_t v^{(N+1)}(t, x) &= 0 && \text{for } t = T_1 \\ v^{(N+1)}(t, x) &= 0 && \text{for } t \geq T_1, \quad x \in \partial\Omega. \end{aligned}$$

The above differential equation has a unique solution. Moreover if  $h$  is the right hand side of (16) we have (see for instance Isakov [14], pp. 185) that if  $T_1 < t < T$  and  $k > 1$

$$\begin{aligned} \|v^{(N+1)}(t, \cdot)\|_{L^2(\Omega)} &\leq C\|h\|_{L^2((T_1, T) \times \Omega)} \\ &\leq \frac{C}{k^N} \end{aligned} \quad (17)$$

Thus, we have shown that we can find a solution  $u = e^{ik(t+\omega \cdot x)}(v_0 + \mathcal{O}(k^{-1}))$  of (1) satisfying the set of initial conditions (2). Let us now examine the first term in (10) by solving the transport equation

$$0 = \mathcal{L}v_0(t, x) = \sum_{j=0}^n \omega_j \partial_{x_j} v_0(t, x) + i \sum_{j=0}^n \omega_j A_j(t, x) v_0(t, x) \quad (18)$$

where we have set  $\omega_0 = 1$  and  $\partial_{x_0} = \partial_t$ .

Equation (18) is a first order transport equation that can be solved by the method of the characteristics or by performing a change of variables that turns the PDE into an ordinary differential equation. Either way, the solution we obtain is given by

$$v_0(t, x) = \chi(t, x) \exp \left( -i \int_{-\infty}^{\frac{1}{2}(t+\omega \cdot x)} \sum_{j=0}^n \omega_j A_j(t' + s, x' + s\omega) ds, \right) \quad (19)$$

where  $(t', x') = (t, x) - \frac{1}{2}(t + \omega \cdot x)(1, \omega)$  is the projection of  $(t, x)$  into  $\Pi_{(1, \omega)}$ , the hyperplane perpendicular to  $(1, \omega)$  and  $\chi$  is any function that is constant



along the direction given by  $(1, \omega)$  and whose support is contained in a neighborhood of the light ray  $\gamma = \{(t', x') + s(1, \omega) \mid s \in \mathbb{R}\}$  (in general  $\chi$  can be complex valued but for our purposes we will assume it is real valued).

Summarizing, we have been able to construct a solution of (1)-(2) having the form:

$$u(t, x) = e^{ik(t-\omega \cdot x) - iR_1(t, x; \omega)} (\chi(t', x') + \mathcal{O}(k^{-1})), \quad (20)$$

where

$$R_1(t, x; \omega) = \int_{-\infty}^{\frac{1}{2}(t+\omega \cdot x)} \sum_{j=0}^n \omega_j A_j(t' + s, x' + s\omega) ds \quad (21)$$

In a similar way one can obtain geometric optics solutions for the backwards hyperbolic problem

$$\begin{aligned} Lv &= 0 & \text{in } (-\infty, T_2) \times \Omega \\ v = \partial_t v &= 0 & \text{for } t = T_2 \end{aligned}$$

In the following section we will derive a Green's Formula for these kinds of hyperbolic operators and will use the Geometric Optics representations to conclude that the Dirichlet to Neumann data determines the vectorial and scalar ray transforms of the potentials along '*light rays*' (this is, rays that make a 45 degree angle with the hyperplane  $t = 0$ ).

## 4 Green's formula

This technique has had a lot of success in the context of inverse problems, in particular for the case of elliptic problems, the fundamental paper of Sylvester and Uhlmann [27] has been a source of inspiration for several other uniqueness results (see also Isakov's review paper [13] for more information on this subject).

For  $T_1$  and  $T_2$  two real numbers with  $T_1 < T_2$  we consider the forward and backward hyperbolic equations

$$\begin{array}{lll} L_1 u = 0 & \text{in } [T_1, T_2] \times \Omega & L_2^* v = 0 \quad \text{in } [T_1, T_2] \times \Omega \\ u = \partial_t u = 0 & \text{for } t = T_1 & v = \partial_t v = 0 \quad \text{for } t = T_2 \\ u = f & \text{on } [T_1, T_2] \times \partial\Omega & v = g \quad \text{on } [T_1, T_2] \times \partial\Omega, \end{array}$$

where we have set

$$L_1 = L(\mathcal{A}^{(1)}, V^{(1)}) = (-i\partial_t + A_0^{(1)}(t, x))^2 - \sum_{j=1}^n (-i\partial_{x_j} + A_j^{(1)}(t, x))^2 + V^{(1)}(t, x)$$

$$L_2^* = L(\overline{\mathcal{A}^{(2)}}, \overline{V^{(2)}}) = (-i\partial_t + \overline{A_0^{(2)}(t, x)})^2 - \sum_{j=1}^n (-i\partial_{x_j} + \overline{A_j^{(2)}(t, x)})^2 + \overline{V^{(2)}(t, x)}$$

and let us assume that the Dirichlet to Neumann operators

$$\Lambda_1(f) = (\partial_\nu + i\nu \cdot A^{(1)}(t, x))u(t, x)|_{[T_1, T_2] \times \partial\Omega} \quad (22)$$

$$\Lambda_2(g) = (\partial_\nu + i\nu \cdot A^{(2)}(t, x))v(t, x)|_{[T_1, T_2] \times \partial\Omega} \quad (23)$$

equal on  $(T_1, T_2) \times \partial\Omega$ , i.e.,  $\Lambda_1 f = \Lambda_2 f$  for all  $f$  smooth and supported on the set  $(T_1, T_2) \times \partial\Omega$ .

**Remark:** Notice that for the operator  $L_2^*$  we associate the Dirichlet to Neumann map

$$\Lambda_2^*(g) = (\partial_\nu + i\nu \cdot \overline{A^{(2)}(t, x)})v(t, x)|_{\mathbb{R} \times \partial\Omega}$$

and that our main assumption is  $\Lambda_1 = \Lambda_2$  on  $\mathbb{R} \times \partial\Omega$ . This is no mistake as later on we will show that our notation is justified as the  $L^2$  adjoint of  $\Lambda_2$  is indeed  $\Lambda_2^*$ .

Denoting by  $\langle \cdot, \cdot \rangle_{[T_1, T_2] \times \Omega}$ ,  $\langle \cdot, \cdot \rangle_\Omega$  and  $\langle \cdot, \cdot \rangle_{[T_1, T_2] \times \partial\Omega}$  the  $L^2$  inner products in  $[T_1, T_2] \times \Omega$ ,  $\Omega$  and  $[T_1, T_2] \times \partial\Omega$  respectively we obtain the following integration by parts formulas for  $A_0^{(1)}$

$$\begin{aligned} \langle (-i\partial_t + A_0^{(1)})^2 u, v \rangle_{[T_1, T_2] \times \Omega} &= \langle (-i\partial_t + A_0^{(1)})u, (-i\partial_t + \overline{A_0^{(1)}})v \rangle_{[T_1, T_2] \times \Omega} \\ &\quad - i \langle (-i\partial_t + A_0^{(1)})u(t, \cdot), v(t, \cdot) \rangle_\Omega \Big|_{T_1}^{T_2} \end{aligned}$$

where  $\nu = (\nu^{(1)}, \dots, \nu^{(n)})$  is the exterior unit normal to  $\partial\Omega$  and  $u, v$  are solutions of the forward and backward hyperbolic equations respectively.

In view of the initial conditions we obtain

$$\langle (-i\partial_t + A_0^{(1)})^2 u, v \rangle_{[T_1, T_2] \times \Omega} = \langle (-i\partial_t + A_0^{(1)})u, (-i\partial_t + \overline{A_0^{(1)}})v \rangle_{[T_1, T_2] \times \Omega}. \quad (24)$$

Also for  $j = 1, \dots, n$  we have for  $A_j^{(1)}$

$$\begin{aligned} \langle (-i\partial_{x_j} + A_j^{(1)})^2 u, v \rangle_{[T_1, T_2] \times \Omega} &= \langle (-i\partial_{x_j} + A_j^{(1)})u, (-i\partial_{x_j} + \overline{A_j^{(1)}})v \rangle_{[T_1, T_2] \times \Omega} \\ &\quad - i \langle (-i\partial_{x_j} + A_j^{(1)})u\nu^{(j)}, v \rangle_{[T_1, T_2] \times \partial\Omega}, \end{aligned}$$

hence

$$\begin{aligned}
\sum_{j=1}^n \langle (-i\partial_{x_j} + A_j^{(1)})^2 u, v \rangle_{[T_1, T_2] \times \Omega} = \\
\sum_{j=1}^n \langle (-i\partial_{x_j} + A_j^{(1)})u, (-i\partial_{x_j} + \overline{A_j^{(1)}})v \rangle_{[T_1, T_2] \times \Omega} \\
- \langle \Lambda_1(f), g \rangle_{[T_1, T_2] \times \partial\Omega}, \quad (25)
\end{aligned}$$

where  $f = u|_{[T_1, T_2] \times \partial\Omega}$  and  $g = v|_{[T_1, T_2] \times \partial\Omega}$ .

Similarly for  $L_{(2)}^*$  we have

$$\langle u, (-i\partial_t + \overline{A_0^{(2)}})^2 v \rangle_{[T_1, T_2] \times \Omega} = \langle (-i\partial_t + A_0^{(2)})u, (-i\partial_t + \overline{A_0^{(2)}})v \rangle_{[T_1, T_2] \times \Omega} \quad (26)$$

and

$$\begin{aligned}
\sum_{j=1}^n \langle u, (-i\partial_{x_j} + \overline{A_j^{(2)}})^2 v \rangle_{[T_1, T_2] \times \Omega} = \\
\sum_{j=1}^n \langle (-i\partial_{x_j} + A_j^{(2)})u, (-i\partial_{x_j} + \overline{A_j^{(2)}})v \rangle_{[T_1, T_2] \times \Omega} \\
- \langle f, \Lambda_2^*(g) \rangle_{[T_1, T_2] \times \partial\Omega}. \quad (27)
\end{aligned}$$

Combining expressions (24)-(27) and recalling that  $u$  and  $v$  are solutions

of the forward and backward hyperbolic problem respectively we obtain

$$\begin{aligned}
0 &= \langle L_1 u, v \rangle_{[T_1, T_2] \times \Omega} - \langle u, L_2^* v \rangle_{[T_1, T_2] \times \Omega} \\
&= \underbrace{\langle (-i\partial_t + A_0^{(1)})u, (-i\partial_t + \overline{A_0^{(1)}})v \rangle_{[T_1, T_2] \times \Omega}}_{I_1} \\
&\quad - \underbrace{\langle (-i\partial_t + A_0^{(2)})u, (-i\partial_t + \overline{A_0^{(2)}})v \rangle_{[T_1, T_2] \times \Omega}}_{I_2} \\
&\quad - \underbrace{\sum_{j=1}^n \langle (-i\partial_{x_j} + A_j^{(1)})u, (-i\partial_{x_j} + \overline{A_j^{(1)}})v \rangle_{[T_1, T_2] \times \Omega}}_{I_3} \\
&\quad + \underbrace{\sum_{j=1}^n \langle (-i\partial_{x_j} + A_j^{(2)})u, (-i\partial_{x_j} + \overline{A_j^{(2)}})v \rangle_{[T_1, T_2] \times \Omega}}_{I_4} \\
&\quad + \langle V^{(1)}u, v \rangle_{[T_1, T_2] \times \Omega} - \langle V^{(2)}u, v \rangle_{[T_1, T_2] \times \Omega} \\
&\quad + \langle f, \Lambda_2^*(g) \rangle_{[T_1, T_2] \times \partial\Omega} - \langle \Lambda_1(f), g \rangle_{[T_1, T_2] \times \partial\Omega} \tag{28}
\end{aligned}$$

Let us now study the terms  $I_1$ ,  $I_2$ ,  $I_3$  and  $I_4$  appearing in the above formula. For  $I_1$  and  $I_2$  we have

$$\begin{aligned}
I_1 + I_2 &= \langle (-i\partial_t + A_0^{(1)})u, (-i\partial_t + \overline{A_0^{(1)}})v \rangle_{[T_1, T_2] \times \Omega} \\
&\quad - \langle (-i\partial_t + A_0^{(2)})u, (-i\partial_t + \overline{A_0^{(2)}})v \rangle_{[T_1, T_2] \times \Omega} \\
&= \langle -i\partial_t u, -i\partial_t v \rangle_{\mathbb{R} \times \Omega} + \langle -i\partial_t u, \overline{A_0^{(1)}}v \rangle_{[T_1, T_2] \times \Omega} + \langle A_0^{(1)}u, -i\partial_t v \rangle_{[T_1, T_2] \times \Omega} \\
&\quad + \langle A_0^{(1)}u, \overline{A_0^{(1)}}v \rangle_{[T_1, T_2] \times \Omega} - \langle -i\partial_t u, -i\partial_t v \rangle_{\mathbb{R} \times \Omega} - \langle -i\partial_t u, \overline{A_0^{(2)}}v \rangle_{[T_1, T_2] \times \Omega} \\
&\quad - \langle A_0^{(2)}u, -i\partial_t v \rangle_{[T_1, T_2] \times \Omega} - \langle A_0^{(2)}u, \overline{A_0^{(2)}}v \rangle_{[T_1, T_2] \times \Omega},
\end{aligned}$$

this is

$$\begin{aligned}
I_1 + I_2 &= \langle (A_0^{(1)} - A_0^{(2)})u, -i\partial_t v \rangle_{[T_1, T_2] \times \Omega} + \langle (A_0^{(1)} - A_0^{(2)})(-i\partial_t u), v \rangle_{[T_1, T_2] \times \Omega} \\
&\quad + \langle ((A_0^{(1)})^2 - (A_0^{(2)})^2)u, v \rangle_{[T_1, T_2] \times \Omega} \tag{29}
\end{aligned}$$

and a similar computation shows that

$$\begin{aligned}
I_2 + I_3 &= \sum_{j=1}^n \langle (A_j^{(1)} - A_j^{(2)})u, -i\partial_{x_j}v \rangle_{[T_1, T_2] \times \Omega} \\
&\quad + \sum_{j=1}^n \langle (A_j^{(1)} - A_j^{(2)})(-i\partial_{x_j}u), v \rangle_{[T_1, T_2] \times \Omega} \\
&\quad + \sum_{j=1}^n \langle ((A_j^{(1)})^2 - (A_j^{(2)})^2)u, v \rangle_{[T_1, T_2] \times \Omega}. \quad (30)
\end{aligned}$$

Combining (28), (29) and (30) we obtain the Green's formula:

$$\begin{aligned}
\langle \Lambda_1(f), g \rangle_{[T_1, T_2] \times \partial\Omega} - \langle f, \Lambda_2^*(g) \rangle_{[T_1, T_2] \times \partial\Omega} &= \\
&\sum_{j=0}^n r_j \left( \langle A_j u, (-i\partial_{x_j}v) \rangle_{[T_1, T_2] \times \Omega} + \langle A_j(-i\partial_{x_j}u), v \rangle_{[T_1, T_2] \times \Omega} \right) \\
&+ \sum_{j=0}^n r_j \langle ((A_j^{(2)})^2 - (A_j^{(1)})^2)u, v \rangle_{[T_1, T_2] \times \Omega} - \langle Vu, v \rangle_{[T_1, T_2] \times \Omega} \quad (31)
\end{aligned}$$

where we have set  $x_0 = t$ ,  $A_j = A_j^{(2)} - A_j^{(1)}$  for  $0 \leq j \leq n$ ,  $V = V^{(2)} - V^{(1)}$ ,  $r_0 = -1$  and  $r_j = 1$  for  $1 \leq j \leq n$ .

At this point it is convenient to notice that if we take the vector and scalar potentials in the forward and backward hyperbolic equation to be the same (i.e.,  $(\mathcal{A}^{(1)}, V^{(1)}) = (\mathcal{A}^{(2)}, V^{(2)})$ ), then we get from (28)

$$\langle \Lambda(f), g \rangle_{[T_1, T_2] \times \partial\Omega} - \langle f, \Lambda^*(g) \rangle_{[T_1, T_2] \times \partial\Omega} = 0$$

proving that  $\Lambda^* = \partial_\nu + i\nu \cdot \overline{A(t, x)}$  is the  $L^2$  adjoint of  $\Lambda = \partial_\nu + i\nu \cdot A(t, x)$ .

Since we are assuming that the Dirichlet to Neumann maps for the forward and backward hyperbolic equations agree ( $\Lambda^1 = \Lambda^2$  on  $\mathbb{R} \times \partial\Omega$ ), then

equation (31) can be rewritten as

$$\begin{aligned}
0 &= \langle \Lambda_1(f), g \rangle_{[T_1, T_2] \times \partial\Omega} - \langle \Lambda_2(f), g \rangle_{[T_1, T_2] \times \partial\Omega} \\
&= \langle \Lambda_1(f), g \rangle_{[T_1, T_2] \times \partial\Omega} - \langle f, \Lambda_2^*(g) \rangle_{[T_1, T_2] \times \partial\Omega} \\
&= \sum_{j=0}^n r_j \left( \langle A_j u, (-i\partial_{x_j} v) \rangle_{[T_1, T_2] \times \Omega} + \langle A_j(-i\partial_{x_j} u), v \rangle_{[T_1, T_2] \times \Omega} \right) \\
&\quad + \sum_{j=0}^n r_j \langle ((A_j^{(2)})^2 - (A_j^{(1)})^2) u, v \rangle_{[T_1, T_2] \times \Omega} - \langle V u, v \rangle_{[T_1, T_2] \times \Omega} \quad (32)
\end{aligned}$$

where as before  $x_0 = t$ ,  $A_j = A_j^{(2)} - A_j^{(1)}$  for  $0 \leq j \leq n$ ,  $V = V^{(2)} - V^{(1)}$ ,  $r_0 = -1$  and  $r_j = 1$  for  $1 \leq j \leq n$ .

## 5 X-ray transform

Our next step is to combine our two main tools, namely the geometric optics representation of the solutions of the forward and backward hyperbolic equations and the Green's formula.

Owing to (20) and (21) we can write geometric optics representations for  $u$  and  $v$  the solutions of the forward and backward hyperbolic equation respectively. These representations are

$$u(t, x) = e^{ik(t-\omega \cdot x) - iR_1(t, x; \omega)} \left( \chi(t', x') + \mathcal{O}(k^{-1}) \right) \quad (33)$$

$$\overline{v(t, x)} = e^{-ik(t-\omega \cdot x) + i\overline{R_2(t, x; \omega)}} \left( \chi(t', x') + \mathcal{O}(k^{-1}) \right), \quad (34)$$

where

$$R_1(t, x; \omega) = \int_{-\infty}^{\frac{1}{2}(t+\omega \cdot x)} \sum_{j=0}^n \omega_j A_j^{(1)}(t' + s, x' + s\omega) ds \quad (35)$$

$$\overline{R_2(t, x; \omega)} = \int_{-\infty}^{\frac{1}{2}(t+\omega \cdot x)} \sum_{j=0}^n \omega_j A_j^{(2)}(t' + s, x' + s\omega) ds. \quad (36)$$

Notice that for  $u$  as in (33), we have for  $0 \leq j \leq n$

$$\begin{aligned}
\partial_{x_j} u &= e^{ik(t-\omega \cdot x) - iR_1(t, x; \omega)} \left( \partial_{x_j} \chi + \mathcal{O}(k^{-1}) + (-ikr_j \omega_j - i\partial_{x_j} R_1) (\chi + \mathcal{O}(k^{-1})) \right) \\
&= k e^{ik(t-\omega \cdot x) - iR_1(t, x; \omega)} \left( -ir_j \omega_j \chi + \mathcal{O}(k^{-1}) \right),
\end{aligned}$$

where we have set  $\omega_0 = 1$  and as before  $r_0 = -1$  and  $r_j = 1$  for  $1 \leq j \leq n$ . Then owing to (34) we obtain

$$(-i\partial_{x_j} u(t, x)) \overline{v(t, x)} = -k e^{i(\overline{R_2(t, x; \omega)} - R_1(t, x; \omega))} (r_j \omega_j \chi(t', x')^2 + \mathcal{O}(k^{-1}))$$

similarly

$$u(t, x) \overline{(-i\partial_{x_j} v(t, x))} = -k e^{i(\overline{R_2(t, x; \omega)} - R_1(t, x; \omega))} (r_j \omega_j \chi(t', x')^2 + \mathcal{O}(k^{-1}))$$

Thus, Green's formula now reads

$$0 = Ck \int_{T_1}^{T_2} \int_{\Omega} \sum_{j=0}^n (A_j^{(2)}(t, x) - A_j^{(1)}(t, x)) r_j^2 \omega_j \chi^2(t', x') \times \\ e^{i(\overline{R_2(t, x; \omega)} - R_1(t, x; \omega))} dx dt + \dots$$

where  $C$  is a (negative) constant and “...” represents terms of order  $\mathcal{O}(1)$ . Dividing the above expression by  $Ck$  and taking the limit as  $k \rightarrow +\infty$  we get

$$0 = \int_{T_1}^{T_2} \int_{\Omega} \sum_{j=0}^n \omega_j (A_j^{(2)}(t, x) - A_j^{(1)}(t, x)) \chi^2(t', x') e^{i(\overline{R_2(t, x; \omega)} - R_1(t, x; \omega))} dx dt.$$

Without loss of generality (cf. Remark 3.1 in [8]) we can assume that  $\text{supp } \mathcal{A}^{(j)} \subset \mathbb{R} \times \Omega$ ,  $j = 1, 2$ . Writing  $X' = (t', x')$  and setting  $\mathcal{A} = (A_0, \dots, A_n) = \mathcal{A}^{(2)} - \mathcal{A}^{(1)}$  we get after the change of variables  $(t, x) = \sigma(1, \omega) + X'$

$$0 = \int_{\Pi(1, \omega)} \int_{-\infty}^{\infty} \sum_{j=0}^n \omega_j A_j(X' + \sigma(1, \omega)) \chi^2(X') e^{i \int_{-\infty}^{\sigma} \sum_{j=0}^n \omega_j A_j(X' + s(1, \omega)) ds} d\sigma dS_{X'}.$$

Since  $\chi$  is an arbitrary function of  $X'$  we then conclude that

$$\begin{aligned} 0 &= \int_{-\infty}^{\infty} \sum_{j=0}^n \omega_j A_j(X' + \sigma(1, \omega)) e^{i \int_{-\infty}^{\sigma} \sum_{j=0}^n \omega_j A_j(X' + s(1, \omega)) ds} d\sigma \\ &= -i \int_{-\infty}^{\infty} \frac{\partial}{\partial \sigma} \left( e^{i \int_{-\infty}^{\sigma} \sum_{j=0}^n \omega_j A_j(X' + s(1, \omega)) ds} \right) d\sigma \\ &= -i \left( e^{i \int_{-\infty}^{\infty} \sum_{j=0}^n \omega_j A_j(X' + s(1, \omega)) ds} - 1 \right) \end{aligned} \tag{37}$$

Summarizing we have proven the following

**Lemma 5.1.** *Suppose that the Dirichlet to Neumann operators  $\Lambda_1$  and  $\Lambda_2$  for the hyperbolic equations*

$$L_k u = \left( (-i\partial_t + A^{(l)}_0(t, x))^2 - \sum_{j=1}^n (-i\partial_{x_j} + A^{(l)}_j(t, x))^2 + V^{(l)}(t, x) \right) u = 0, \quad k = 1, 2$$

*equal on  $[T_1, T_2] \times \partial\Omega$ . Then for any light ray*

$$\gamma = \{(t', x') + s(1, \omega) : s \in \mathbb{R}, \omega \in S^{n-1}, (t', x') \cdot (1, \omega) = 0\},$$

*the vectorial ray transform of  $\mathcal{A} = (A_0^{(2)} - A_0^{(1)}, \dots, A_n^{(2)} - A_n^{(1)})$  along  $\gamma$  is an integer multiple of  $2\pi$ . This is*

$$(\mathcal{PA})(t, x; \omega) := \int_{-\infty}^{\infty} \sum_{j=0}^n \omega_j A_j(t' + s, x' + s\omega) ds = 2\pi r \quad (38)$$

*for some  $r \in \mathbb{Z}$ . Here  $(t', x') = (t, x) - \frac{1}{2}(t + \omega \cdot x)(1, \omega)$  and  $\omega_0 = 1$ .*

*Proof.* Equation (37) can be rewritten as

$$e^{i(\mathcal{PA})(t, x; \omega)} = 1$$

which in turn implies (38).  $\square$

If we now incorporate the hypothesis of  $\mathcal{A}^{(1)}$  and  $\mathcal{A}^{(2)}$  being compactly supported in  $x$  we can determine the exact value of  $r$ . This is because equation (38) holds for any  $(t, x; \omega) \in \mathbb{R}_{t,x}^{m+1} \times S^{m-1}$  and in particular, when  $t = 0$  and  $|x|$  is big enough and perpendicular to a fixed  $\omega$ , the light ray  $(0, x) + s(1, \omega)$ ,  $s \in \mathbb{R}$  does not meet the support of  $\mathcal{A}$ , hence

$$\int_{-\infty}^{\infty} \sum_{j=0}^m \omega_j A_j(t' + s, x' + s\omega) ds = 0. \quad (39)$$

To conclude this section let us proceed to remove the condition  $(t', x') \cdot (1, \omega) = 0$ . If  $(t, x)$  is an arbitrary point in  $\mathbb{R}_{t,x}^{n+1}$ , then by making the change of variables  $s = \sigma - \frac{t + \omega \cdot x}{2}$  we get

$$\int_{-\infty}^{\infty} \sum_{j=0}^n \omega_j A_j(t + \sigma, x + \sigma\omega) d\sigma = \int_{-\infty}^{\infty} \sum_{j=0}^n \omega_j A_j(t' + s, x' + s\omega) ds,$$

where  $(t', x') = (t, x) - \frac{(t + \omega \cdot x)}{2}(1, \omega)$ . Clearly this last integral equals zero by (39).



## 6 The main theorem

In this section we will establish several uniqueness results for vector and scalar potentials satisfying different growth conditions. Let us proceed first with the part that deals with the vector potentials in the case when the component functions  $A_j(t, x)$  decay exponentially in  $t$ .

**Theorem 6.1.** *Suppose that  $\mathcal{A}(t, x) = (A_0(t, x), \dots, A_n(t, x))$  with  $\mathcal{A} \in C^\infty$  in  $x$  and  $t$  is such that for any non-negative integers  $\alpha, \beta$  and for any  $0 \leq j \leq n$  there exist positive constants  $c, C_{\alpha, \beta}$  such that for  $|t| \geq t_0$ ,  $|\partial_t^\alpha \partial_x^\beta A_j(t, x)| \leq C_{\alpha, \beta} e^{-c|t|}$ . If in addition  $A_j(t, x) = 0$  for  $|x| \geq R > 0$  and (39) holds, then there exists  $\varphi(t, x) \in C^\infty(t, x)$  and positive constants  $C'_{\alpha, \beta}, c'$  such that*

$$i) \quad A_0(t, x) = \partial_t \varphi(t, x), \quad A_j(t, x) = \partial_{x_j} \varphi(t, x), \quad 1 \leq j \leq n, \text{ and}$$

$$ii) \quad \text{Supp } \varphi \subseteq \mathbb{R} \times \{|x| \leq R\}, \quad |\partial_t^\alpha \partial_x^\beta \varphi(t, x)| \leq C'_{\alpha, \beta} e^{-c'|t|}.$$

*Proof.* By uniqueness of the Fourier transform and by (39) we have

$$0 = \iint e^{-it\tau - ix \cdot \xi} \left( \int_{-\infty}^{\infty} \sum_{j=0}^n \omega_j A_j(t+s, x+s\omega) ds \right) dt dx.$$

By the hypothesis on the support of  $A_j$ ,  $1 \leq j \leq n$ , we can change the order of integration. After writing  $t_1 = t+s$ ,  $x_1 = x+s\omega$ , we are lead to

$$\begin{aligned} 0 &= \iiint e^{-i(t_1-s)\tau - i(x_1-s\omega) \cdot \xi} \left( A_0 + \sum_{j=1}^n \omega_j A_j \right) (t_1, x_1) dt_1 dx_1 ds \\ &= \int e^{is(\tau + \omega \cdot \xi)} \left( A_0 + \sum_{j=1}^n \omega_j A_j \right)^\wedge (\tau, \xi) ds \\ &= \delta(\tau + \omega \cdot \xi) \left( A_0 + \sum_{j=1}^n \omega_j A_j \right)^\wedge (\tau, \xi), \end{aligned} \tag{40}$$

which tells us that the Fourier transform of  $A_0 + \sum_{j=1}^n \omega_j A_j$  vanishes on  $\Pi_{(1, \omega)}$ , the hyperplane perpendicular to  $(1, \omega)$ .

We claim that this Fourier transform vanishes in the complement of the 'light cone'  $\mathcal{C} = \{(\tau, \xi) : |\tau| \geq |\xi|\}$  for an appropriate choice of  $\omega$ . To see

this notice that if  $(\tau, \xi) \notin \mathcal{C}$ , then  $\frac{-\tau}{|\xi|}$  has norm less than one and we can find  $\omega = \omega(\tau, \xi) \in S^{n-1}$  satisfying

$$\frac{\xi}{|\xi|} \cdot \omega(\tau, \xi) = -\frac{\tau}{|\xi|}. \quad (41)$$

Clearly with this choice of  $\omega$  we have  $\tau + \omega(\tau, \xi) \cdot \xi = 0$  and the function  $(A_0 + \sum_{j=1}^n \omega_j(\tau, \xi) A_j)^\wedge(\tau, \xi)$  vanishes when  $|\tau| < |\xi|$ . Moreover, this shows that the Fourier transform of the vector potential  $\widehat{\mathcal{A}}(\tau, \xi)$  is perpendicular to the  $(n+1)$ -dimensional vector  $(1, \omega(\tau, \xi))$  as

$$(A_0 + \sum_{j=1}^n \omega_j(\tau, \xi) A_j)^\wedge(\tau, \xi) = (1, \omega(\tau, \xi)) \cdot \widehat{\mathcal{A}}(\tau, \xi). \quad (42)$$

Equation (41) has infinitely many solutions and as a matter of fact they can be parametrized by  $S^{n-2}$ . On the other hand, equation (42) tells us that  $\widehat{\mathcal{A}} = (\widehat{A}_0, \dots, \widehat{A}_n)$  is orthogonal to all elements of  $E = \{(1, \omega(\tau, \xi)) : \tau + \omega(\tau, \xi) \cdot \xi = 0\}$ . It is not hard to prove (see Appendix A) that the orthogonal complement  $E^\perp$  is one dimensional and since  $(\tau, \xi)$  is perpendicular to any vector of the form  $(1, \omega(\tau, \xi))$ , this complement has to agree with the line  $\{c(\tau, \xi) : c \in \mathbb{R}\}$ .

Since the previous argument works for an arbitrary  $\tau$  and since the set  $\{\xi : |\tau| < |\xi|\}$  is an open subset in  $\mathbb{R}^n$ , we see that  $\widehat{\mathcal{A}}(\tau, \xi) = (\widehat{A}_0(\tau, \xi), \dots, \widehat{A}_n(\tau, \xi))$  is proportional to the vector  $(\tau, \xi)$  in the complement of the light cone. In other words, we can find a function  $\Phi$  such that

$$(\widehat{A}_0(\tau, \xi), \dots, \widehat{A}_n(\tau, \xi)) = i\Phi(\tau, \xi) (\tau, \xi) \quad (43)$$

whenever  $|\tau| < |\xi|$ . Since for any  $j$  the function  $A_j$  decays exponentially in  $t$  and is compactly supported in  $x$  then its Fourier transform  $\widehat{A}_j$  is analytic in the strip  $|\operatorname{Im} \tau| < c$ .

On the other hand, equation (43) gives

$$\begin{aligned} \Phi(\tau, \xi) &= -\frac{i\widehat{A}_j(\tau, \xi)}{\xi^{(j)}}, \quad 1 \leq j \leq n, \\ \Phi(\tau, \xi) &= -\frac{i\widehat{A}_0(\tau, \xi)}{\tau}, \end{aligned}$$

which tell us that  $\Phi$  is analytic in the set  $\{(\tau, \xi) : |\operatorname{Im} \tau| < c, (\tau, \xi) \neq (0, 0)\}$ . Hartog's theorem (see [11]) tells us that the concepts of removable singularities and isolated singularities agree in functions of several complex variables and we conclude that  $\Phi$  is analytic in the strip  $|\operatorname{Im} \tau| < c$ . Moreover if we let  $\varphi$  be the inverse Fourier transform of  $\Phi$ , then  $\varphi$  and all of its derivatives are exponentially decaying in  $t$  and we only need to make sure that it has the right support properties.

Because of the assumptions on the support of the functions  $A_j$  we have, by the Paley-Wiener theorem

$$|\widehat{A}_j(\tau, \xi)| \leq \frac{C_N^{(j)} \exp(R|\operatorname{Im} \xi|)}{(1 + |\xi|)^N}, \quad 0 \leq j \leq n,$$

for  $1 \leq j \leq n$ . Then for  $|\xi^{(j)}| > 1$

$$|\Phi(\tau, \xi)| = \left| \frac{\widehat{A}_j(\tau, \xi)}{\xi^{(j)}} \right| \leq \frac{C_N^{(j)} \exp(R|\operatorname{Im} \xi|)}{(1 + |\xi|)^N}, \quad 0 \leq j \leq n,$$

for some  $C_N > 0$ . Since the function  $h(\tau, \xi) = \Phi(\tau, \xi)(1 + |\xi|)^N \exp(-R|\operatorname{Im} \xi|)$  is continuous when  $|\xi^{(j)}| \leq 1$ , it is also bounded, hence  $|h(\tau, \xi)| \leq C_N$  for some positive  $C_N$  and the estimate

$$|\Phi(\tau, \xi)| \leq \frac{C \exp(R|\operatorname{Im} \xi|)}{(1 + |\xi|)^N} \quad (44)$$

holds for any  $\xi \in \mathbb{R}^n$ . Making use once again of the Paley-Wiener theorem we conclude that the inverse fourier transform of  $\Phi(\tau, \xi)$  is supported in the set  $\{x : |x| \leq R\}$ .  $\square$

Before going ahead to prove the corresponding equality of the scalar potentials, we will pause for a second to relax the conditions imposed on the vector potential. Let us start by replacing exponentially decaying by Schwartz functions.

**Theorem 6.2.** *Suppose that  $\mathcal{A}(t, x) = (A_0(t, x), \dots, A_n(t, x))$  with  $\mathcal{A} \in C^\infty$  in  $x$  and  $t$  is such that for any  $M > 0$  and non-negative integers  $\alpha, \beta$  there exist constants  $C_{M, \alpha, \beta} > 0$  such that  $(1 + |t|)^M |\partial_t^\alpha \partial_x^\beta A_j(t, x)| \leq C_{M, \alpha, \beta}$  for  $0 \leq j \leq n$ . If in addition  $A_j(t, x) = 0$  for  $|x| \geq R > 0$  and (39) holds, then there exists  $\varphi(t, x) \in C^\infty(t, x)$  such that*

i)  $A_0(t, x) = \partial_t \varphi(t, x)$ ,  $A_j(t, x) = \partial_{x_j} \varphi(t, x)$ ,  $1 \leq j \leq n$ , and

ii)  $\text{Supp } \varphi \subseteq \mathbb{R} \times \{|x| \leq R\}$ ,  $(1 + |t|)^M |\partial_t^\alpha \partial_x^\beta \varphi(t, x)| \leq C'_{M, \alpha, \beta}$ .

*Proof.* The proof goes along the same lines as the previous proposition except that now in equation (43) we only know that the left hand side is entire in  $\xi$ . For  $\tau_0 \neq 0$  fixed, Hartog's theorem tells us that  $\Phi(\tau_0, \xi)$  is entire and when  $\tau = 0$ , equation (43) gives  $\Phi(0, \xi) = -\frac{i\widehat{A_j}(0, \xi)}{\xi^{(j)}}$  for  $1 \leq j \leq n$  showing that  $\Phi$  has no singularities.

The part of the proof that deals with the support of  $\varphi$  remains unchanged and we only need to show that  $\Phi$  is a Schwartz function. If  $M' > 0$  and  $\beta$  is a non-negative integer, we have for  $|\xi| \leq R$

$$\begin{aligned} (1 + |\tau|)^{\tilde{M}} |\partial_\tau^\beta \Phi(\tau, \xi)| &= (1 + |\tau|)^{\tilde{M}} \left| \partial_\tau^\beta \left( i \frac{\widehat{A_0}(\tau, \xi)}{\tau} \right) \right| \\ &= (1 + |\tau|)^{\tilde{M}} \left| \sum_{j=0}^{\beta} c_j \partial_\tau^j \widehat{A_0}(\tau, \xi) \partial_\tau^{\beta-j} \left( \frac{1}{\tau} \right) \right| \\ &\leq C(1 + |\tau|)^{\tilde{M}'} \left| \sum_{j=0}^{\beta} \partial_\tau^j \widehat{A_0}(\tau, \xi) \right| \leq C_{\tilde{M}', \beta, R}, \end{aligned}$$

where in the last inequality we used the fact that the exponentially decaying  $C^\infty$  functions Fourier transform into Schwartz functions. Since  $\Phi$  is itself Schwartz, the desired function  $\varphi$  is again the inverse Fourier transform of  $\Phi$ .  $\square$

The conditions on the vector potential imposed so far are such that we end up working with functions once we compute the Fourier transform of equation (39), nevertheless, this transform can be computed under weaker assumptions and the following theorem tells us that the final result is still valid.

**Theorem 6.3.** *Suppose that  $\mathcal{A}(t, x) = (A_0(t, x), \dots, A_n(t, x))$  with  $\mathcal{A} \in C^\infty$  in  $x$  and  $t$  is such that for  $0 \leq j \leq n$ ,  $|A_j(t, x)| \leq C(1 + |t|)^M$  with  $C, M > 0$  and  $|t| \geq t_0$ . If in addition the functions  $|A_j(t, x)|$  are locally integrable in  $\mathbb{R}^{n+1}$ , satisfy the support condition  $A_j(t, x) = 0$  for  $|x| \geq R > 0$ , and equation (39) holds; then there exists  $\varphi(t, x) \in C^\infty(t, x)$  such that*

i)  $A_0(t, x) = \partial_t \varphi(t, x)$ ,  $A_j(t, x) = \partial_{x_j} \varphi(t, x)$ ,  $1 \leq j \leq n$ , and

ii)  $\text{Supp } \varphi \subseteq \mathbb{R} \times \{|x| \leq R\}$ .

*Proof.* By the hypothesis on the growth of  $A_j$ ,  $1 \leq j \leq n$ , we can compute the Fourier transform of equation (39) to obtain  $\delta(\tau + \omega \cdot \xi)(A_0 + \sum_{j=1}^n \omega_j A_j)^\wedge(\tau, \xi) = 0$ , where  $\hat{A}_j(\tau, \xi)$  is an analytic function in  $\xi$  and a distribution in  $\tau$ . In addition, since the wavefront set of  $\delta(\tau + \omega \cdot \xi)$  and  $\hat{A}_0 + \sum_{j=1}^n \omega_j \cdot \hat{A}_j$  do not intersect, we can define a restriction of  $\hat{A}_0 + \sum_{j=1}^n \omega_j \cdot \hat{A}_j$  on the hyperplane  $\tau + \omega \cdot \xi = 0$  (cf. Hörmander [12]). Proceeding as before we find that when  $|\tau| < |\xi|$  there are infinitely many solutions of equation (41) and that they can be parametrized by  $S^{n-2}$ . Moreover, the change  $(\tau, \xi) \rightarrow (\alpha\tau, \alpha\xi)$ ,  $\alpha > 0$ , in (41) leads to

$$\begin{aligned} \frac{\alpha\xi}{|\alpha||\xi|} \cdot \omega(\alpha\tau, \alpha\xi) &= -\frac{\alpha\tau}{|\alpha||\tau|} \\ \frac{\xi}{|\xi|} \cdot \omega(\alpha\tau, \alpha\xi) &= -\frac{\tau}{|\tau|}, \end{aligned}$$

which tells us that the solutions  $\omega(\tau, \xi)$  of (41) are homogeneous of degree 0 in  $(\tau, \xi)$ .

Therefore

$$\hat{A}_0(\tau, \xi) + \sum_{j=1}^n \omega_j \hat{A}_j(\tau, \xi) = 0 \quad (45)$$

on the plane  $\tau + \omega \cdot \xi = 0$ .

Replace  $(\tau, \xi)$  by  $(\alpha\tau, \alpha\xi)$  where  $\alpha > 0$  and we then see that for  $\alpha > 0$

$$\hat{A}_0(\alpha\tau, \alpha\xi) + \sum_{j=1}^n \omega_j \hat{A}_j(\alpha\tau, \alpha\xi) = 0. \quad (46)$$

We now let  $\chi(\alpha)$  be an arbitrary  $C_0^\infty(\mathbb{R})$  function with support contained in the set  $|\alpha - 1| < \epsilon$  and multiply (46) by  $\chi(\alpha)$ . Integration in  $\alpha$  leads to

$$a_0(\tau, \xi) + \sum_{j=1}^n \omega_j a_j(\tau, \xi) = 0, \quad (47)$$

where  $\tau + \omega \cdot \xi = 0$  and

$$a_j(\tau, \xi) = \int_{-\infty}^{\infty} \hat{A}_j(\alpha\tau, \alpha\xi) \chi(\alpha) d\alpha.$$

Notice that  $a_j(\tau, \xi)$  are no longer distributions and we can put  $\omega = \omega(\tau, \xi)$  in (47). Arguing as before we find that  $(a_0(\tau, \xi), \dots, a_n(\tau, \xi)) = ib(\tau, \xi)(\tau, \xi)$  for some  $b(\tau, \xi)$ , or in other words,

$$\frac{a_0(\tau, \xi)}{\tau} = \frac{a_1(\tau, \xi)}{\xi_1} = \dots = \frac{a_n(\tau, \xi)}{\xi_n} = ib(\tau, \xi).$$

Since  $\chi(\alpha)$  is arbitrary we get

$$\frac{\widehat{A}_0(\alpha\tau, \alpha\xi)}{\alpha\tau} = \frac{\widehat{A}_1(\alpha\tau, \alpha\xi)}{\alpha\xi_1} = \dots = \frac{\widehat{A}_n(\alpha\tau, \alpha\xi)}{\alpha\xi_n} = i\widehat{\Psi}(\alpha\tau, \alpha\xi),$$

where  $\widehat{\Psi}(\alpha\tau, \alpha\xi)$  is a distribution in  $\alpha\tau$  for all  $\alpha \in (1-\epsilon, 1+\epsilon)$ . Finally when  $\alpha = 1$  we get

$$\widehat{A}_0(\tau, \xi) = i\tau\widehat{\Psi}(\tau, \xi), \quad \widehat{A}_1(\tau, \xi) = i\xi_1\widehat{\Psi}(\tau, \xi), \quad \dots, \quad \widehat{A}_n(\tau, \xi) = i\xi_n\widehat{\Psi}(\tau, \xi)$$

for  $|\tau| < |\xi|$ . As before we have that  $\widehat{\Psi}$  is entire in  $\xi$  and  $\widehat{\Psi} \in S'$  in  $\tau$  (since  $A_j \in S'$  in  $\tau$ ). Therefore  $\varphi = \mathcal{F}_{\tau, \xi}^{-1}\widehat{\Psi} \in S'$  in  $t$ . Moreover the identities  $\partial_t\varphi = A_0, \partial_{x_1}\varphi = A_1, \dots, \partial_{x_n}\varphi = A_n$  imply that  $\varphi(t, x) \in C^\infty$  in  $(t, x)$  and that  $\varphi = 0$  for  $|x| > R$ .  $\square$

Summarizing, the three previous results prove that the vector potentials  $\mathcal{A}^{(1)}$  and  $\mathcal{A}^{(2)}$  are gauge equivalent with gauge  $g = e^{i\varphi}$ . Next, we show that this equivalence implies the equality of the scalar potentials  $V^{(1)}$  and  $V^{(2)}$ .

By the previous proposition  $\mathcal{A}^{(2)} - \mathcal{A}^{(1)} = \nabla_{t,x}\varphi$ , replacing the pair  $(\mathcal{A}^{(1)}, V^{(1)})$  by  $(\mathcal{A}^{(3)}, V^{(3)})$  where  $\mathcal{A}^{(3)} = \mathcal{A}^{(1)} + \nabla_{t,x}\varphi$  and  $V^{(3)} = V^{(1)}$ , we find by means of proposition 2.1 that  $\mathcal{A}^{(3)} = \mathcal{A}^{(2)}$ . Next we use our Green's formula (32) with the pair of potentials  $(\mathcal{A}^{(2)}, V^{(2)})$  and  $(\mathcal{A}^{(3)}, V^{(3)})$  to obtain

$$0 = \langle (V^{(3)} - V^{(2)})u, v \rangle_{[T_1, T_2] \times \Omega} = \int_{T_1}^{T_2} \int_{\Omega} (V^{(3)} - V^{(2)}) u \bar{v} dx dt.$$

Making use of the geometric optics representations (33)-(36) the above integral becomes

$$0 = \int_{T_1}^{T_2} \int_{\Omega} (V^{(3)}(t, x) - V^{(2)}(t, x)) e^{i(\overline{R_2(t, x; \omega)} - R_1(t, x; \omega))} \chi^2(t', x') dx dt + \dots \quad (48)$$

where  $\dots$  denotes terms of order  $\mathcal{O}(k^{-1})$ . Taking the limit as  $k \rightarrow +\infty$  we notice that the equality of the vector potentials imply that  $\overline{R_2} - R_1 = 0$ . Thus, after a change of variables, we can rewrite (48) as

$$0 = \int_{\Pi_{(1,\omega)}} \left( \int_{-\infty}^{\infty} V^{(3)}(X' + s(1, \omega)) - V^{(2)}(X' + s(1, \omega)) ds \right) \chi^2(X') dS_{X'},$$

since  $\chi$  is arbitrary the inner integral in the expression above vanishes

$$\int_{-\infty}^{\infty} (V^{(3)}(t' + s, x' + s\omega) - V^{(2)}(t' + s, x' + s\omega)) ds = 0 \quad (49)$$

which shows that the light ray transform of the potentials agree. A simple variation of the previous proof applies and we have  $V^{(1)} = V^{(3)} = V^{(2)}$ . Therefore the pair of potentials  $(\mathcal{A}^{(1)}, V^{(1)})$  and  $(\mathcal{A}^{(2)}, V^{(2)})$  are gauge equivalent (see also [25],[23]).

## 7 Stability estimate

In this section we assume that the components of the vector potentials  $\mathcal{A}^{(1)}$  and  $\mathcal{A}^{(2)}$  are real valued, smooth and compactly supported in both  $t$  and  $x$ . Just as we did before, let us write

$$\mathcal{A} = \mathcal{A}^{(1)} - \mathcal{A}^{(2)} \quad \text{where} \quad \mathcal{A}^{(k)} = (A_0^{(k)}, \dots, A_n^{(k)}), \quad k = 1, 2,$$

and let us further assume that the potential  $\mathcal{A}$  satisfies the divergence condition

$$\operatorname{div} \mathcal{A} = \partial_t A_0(t, x) + \sum_{j=1}^n \partial_{x_j} A_j(t, x) = 0. \quad (50)$$

Since the potentials are compactly supported we can find real numbers  $T_1 < 0 < T_2$  and an open set  $\mathcal{D} \subseteq \mathbb{R}_t \times \mathbb{R}_x^n$  such that if we set  $Q = (T_1, T_2) \times \Omega$ , then

i)  $Q \subseteq \mathcal{D}$ ; and

ii)  $\operatorname{Supp}(A_j) \subseteq \mathcal{D}$  for  $j = 0, 1, \dots, n$ .

In other words,  $\mathcal{D}$  is a set containing  $Q$  and the support of all the components of the vector potential. Since  $\mathcal{D}$  is bounded we can find  $T_3$  bigger than  $|T_1|$

and  $|T_2|$  such that  $\mathcal{D} \subseteq (-T, T) \times \mathbb{R}_x^n$ , we then choose  $T > T_3 + \text{diam}(\Omega)$  and set  $Q_T := (-T, T) \times \Omega$ .

When  $T$  is selected appropriately we can find solutions

$$u(t, x) = \chi_\omega(t, x) e^{ik(t-\omega \cdot x) + iR_1(t, x; \omega)} (1 + \mathcal{O}(k^{-1}))$$

and

$$v(t, x) = \chi_\omega(t, x) e^{ik(t-\omega \cdot x) + i\overline{R_2(t, x; \omega)}} (1 + \mathcal{O}(k^{-1}))$$

of the backward and forward hyperbolic equations satisfying

$$\begin{aligned} u = u_t = 0 & \quad \text{on} \quad \{-T\} \times \Omega, \\ v = v_t = 0 & \quad \text{on} \quad \{T\} \times \Omega, \end{aligned}$$

where the function  $\chi_\omega$  is such that

- a)  $(\partial_t + \sum_{j=1}^n \omega_j \partial_{x_j}) \chi_\omega(t, x) = 0$ . This is,  $\chi_\omega$  is constant along light rays;  
and
- b)  $\chi_\omega$  is supported in a small neighborhood of the ray  $\{(t + s, x + s\omega) : s \in \mathbb{R}\}$ .

As before we can make use of the Green's formula developed in previous sections to obtain

$$\begin{aligned} \langle (\Lambda_1 - \Lambda_2)(f), g \rangle_{(-T, T) \times \partial\Omega} = I_{Q_T} := & \sum_{j=1}^n (\langle A_j u, -i \partial_{x_j} v \rangle_{Q_T} + \langle A_j (-i \partial_{x_j} u), v \rangle_{Q_T}) \\ & + \sum_{j=1}^n \langle ((A_j^{(2)})^2 - (A_j^{(1)})^2) u, v \rangle_{Q_T} - \langle V u, v \rangle_{Q_T} \\ & - \langle A_0 u, -i \partial_t v \rangle_{Q_T} - \langle A_0 (-i \partial_t u), v \rangle_{Q_T} \\ & - \langle ((A_0^{(2)})^2 - (A_0^{(1)})^2) u, v \rangle_{Q_T} \end{aligned} \tag{51}$$

where we have set

$$f = u(t, x)|_{(-T, T) \times \partial\Omega} \quad g = v(t, x)|_{(-T, T) \times \partial\Omega}.$$



We next regard  $\Lambda_1 - \Lambda_2$  as a map from  $H^1 \rightarrow H^0$  and denoting by  $||| \quad |||$  the operator norm between these spaces, we have by the Cauchy-Schwarz inequality

$$|I_{Q_T}| = \left| \langle (\Lambda_1 - \Lambda_2)(f), g \rangle_{(-T, T) \times \partial\Omega} \right| \leq ||| \Lambda_1 - \Lambda_2 ||| \times \\ \|f\|_{H^1((-T, T) \times \partial\Omega)} \|g\|_{L^2((-T, T) \times \partial\Omega)}.$$

The latter norm can be estimated by

$$\begin{aligned} \|g\|_{L^2((-T, T) \times \partial\Omega)} &= \|\chi_\omega(t, x)(1 + \mathcal{O}(k^{-1}))\|_{L^2((-T, T) \times \partial\Omega)} \\ &\leq \|\chi_\omega(t, x)\|_{L^2((-T, T) \times \partial\Omega)} + \mathcal{O}(k^{-1}), \end{aligned} \quad (52)$$

whereas the middle norm can be estimated by

$$\begin{aligned} \|f\|_{H^1((-T, T) \times \partial\Omega)} &\leq C(n, \Omega) \left[ k \|\chi_\omega\|_{L^2((-T, T) \times \partial\Omega)} + \mathcal{O}(1) \right] \\ &= C(n, \Omega) k \left[ \|\chi_\omega\|_{L^2((-T, T) \times \partial\Omega)} + \mathcal{O}(k^{-1}) \right]. \end{aligned} \quad (53)$$

If in addition we assume that  $\|\chi_\omega\|_{L^2((-T, T) \times \partial\Omega)} \leq C$  we have by (52) and (53)

$$|I_{Q_T}| \leq kC(n, \Omega) \left[ ||| \Lambda_1 - \Lambda_2 ||| + \mathcal{O}(k^{-1}) \right], \quad (54)$$

where  $C(n, \Omega)$  is a constant that depends upon the supremum of all ray integrals of  $A_0 + \sum_{j=1}^n \omega_j A_j$ . On the other hand, the integral  $I_{Q_T}$  leads, just as before, to

$$\left| Ck \int_{-T}^T \int_{\Omega} \left( A_0 + \sum_{j=1}^n \omega_j A_j \right) (t, x) \chi_\omega^2(t, x) \times \right. \\ \left. e^{-i \int_{-\infty}^{\frac{1}{2}(t+\omega \cdot x)} \left( A_0 + \sum_{j=1}^n \omega_j A_j \right) (t' + s, x' + s\omega) ds} dx dt + \dots \right|$$

where “ $\dots$ ” represents terms that, when divided by  $k$ , go to zero as  $k \rightarrow +\infty$ .

Dividing the above expression by  $k$  and taking the limit as  $k \rightarrow \infty$  we get, after using the triangle inequality and performing the change of coordinates

$(t, x) = \sigma(1, \omega) + Y'$  with  $Y' \in \Pi_{(1, \omega)}$

$$\left| \int_{\Pi_{(1, \omega)}} \int_{\mathbb{R}} \left( A_0 + \sum_{j=1}^n \omega_j A_j \right) (Y' + \sigma(1, \omega)) \chi_{\omega}^2(Y') \times \right. \\ \left. e^{-i \int_{-\infty}^{\sigma} (A_0 + \sum_{j=1}^n \omega_j A_j) (Y' + s(1, \omega)) ds} d\sigma dS_{Y'} \right| \leq C(n, \Omega) |||\Lambda_1 - \Lambda_2|||. \quad (55)$$

If we set

$$a(Y') := \int_{\mathbb{R}} \left( A_0 + \sum_{j=1}^n \omega_j A_j \right) (Y' + \sigma(1, \omega)) e^{-i \int_{-\infty}^{\sigma} (A_0 + \sum_{j=1}^n \omega_j A_j) (Y' + s(1, \omega)) ds} d\sigma,$$

equation (55) can be rewritten as

$$\left| \int_{\Pi_{(1, \omega)}} a(Y') \chi^2(Y') dS_{Y'} \right| \leq C(n, \Omega) |||\Lambda_1 - \Lambda_2|||.$$

On the other hand the conditions imposed on the support of  $\chi$  guarantee that the above estimate holds true for any such function satisfying the condition  $\int_{\Pi_{(1, \omega)}} |\chi(Y')|^2 dS_{Y'} \leq 1$ , thus  $a$  is a bounded linear functional on  $L^1(\Pi_{(1, \omega)})$  and the estimate

$$\left| \int_{-\infty}^{\infty} \left( A_0 + \sum_{j=1}^n \omega_j A_j \right) (X' + \sigma(1, \omega)) \times \right. \\ \left. e^{i \int_{-\infty}^{\sigma} (A_0 + \sum_{j=1}^n \omega_j A_j) (X' + s(1, \omega)) ds} d\sigma \right| \leq C(n, \Omega) |||\Lambda_1 - \Lambda_2|||$$

holds. The Fundamental Theorem of Calculus then gives (in the original coordinate system)

$$\left| \exp \left[ i \int_{-\infty}^{\infty} \left( A_0 + \sum_{j=1}^n \omega_j A_j \right) (t + s, x + s\omega) ds \right] - 1 \right| \leq C(n, \Omega) |||\Lambda_1 - \Lambda_2|||. \quad (56)$$

In the case of uniqueness of the potentials it was very easy to go from an expression concerning the above complex exponential to an expression involving only the ray transform of the function  $A_0 + \sum_{j=1}^n \omega_j A_j$ . In this case, obtaining such an estimate is slightly harder and we need to assume that the following condition holds:

iii) the supremum

$$\alpha := \sup_{(t,x;\omega) \in Q \times S^{n-1}} \left| \int_{-\infty}^{\infty} (A_0 + \sum_{j=1}^n \omega_j A_j)(t+s, x+s\omega) ds \right|$$

satisfies the inequality  $\alpha < 2\pi$ .

Denoting by  $\beta$  the integral  $\int_{-\infty}^{\infty} (A_0 + \sum_{j=1}^n \omega_j A_j)(t+s, x+s\omega) ds$  we get

$$\frac{|e^{i\beta} - 1|}{|\beta|} = \frac{|\sin \frac{\beta}{2}|}{\frac{|\beta|}{2}}. \quad (57)$$

Also, condition iii) gives  $\frac{|\beta|}{2} < \frac{\alpha}{2} < \pi$  and the right hand side of (57) is bounded from below by a positive constant  $\frac{1}{C_4}$ . We then have the estimate

$$\left| \int_{-\infty}^{\infty} (A_0 + \sum_{j=1}^n \omega_j A_j)(t+s, x+s\omega) ds \right| \leq C_4 \left| e^{i \int_{-\infty}^{\infty} (A_0 + \sum_{j=1}^n \omega_j A_j)(t+s, x+s\omega) ds} - 1 \right|,$$

which together with (56) gives

$$\left| \int_{-\infty}^{\infty} (A_0 + \sum_{j=1}^n \omega_j A_j)(t+s, x+s\omega) ds \right| \leq C(n, \Omega) |||\Lambda_1 - \Lambda_2|||. \quad (58)$$

We next want to use (58) as well as the divergence condition imposed on the potentials to obtain an estimate for the potentials  $A_j$ ,  $j = 0, \dots, n$ , following the ideas in Begmatov's paper [2].

If we let  $F$  denote the ray transform of  $A_0 + \sum_{j=1}^n \omega_j A_j$  along light rays, we have

$$F : \mathbb{R}_t \times \mathbb{R}_x^n \times S^{n-1} \rightarrow \mathbb{R}$$

$$F(t, x; \omega) := \int_{\mathbb{R}} (A_0 + \sum_{j=1}^n \omega_j A_j)(t+s, x+s\omega) ds. \quad (59)$$

and by (58)

$$|F(t, x; \omega)| \leq C(n, \Omega) |||\Lambda_1 - \Lambda_2||| \quad (60)$$

for all  $(t, x) \in \mathbb{R}_t \times \mathbb{R}_x^n$ ,  $\omega \in S^{n-1}$ .

The Fourier transform of  $F$  in the variables  $x_1, \dots, x_n$  is

$$(\mathcal{F}_{(x \rightarrow \xi)} F(t, \cdot; \omega))(\xi) = \int_{\mathbb{R}^n} e^{-i\xi \cdot x} \int_{\mathbb{R}} \left( A_0 + \sum_{j=1}^n \omega_j A_j \right) (t + s, x + s\omega) \, ds \, dx.$$

and the change of coordinates  $\tilde{x} = x + s\omega$ ,  $\tilde{t} = t + s$ , with Jacobian  $\left| \frac{\partial(\tilde{t}, \tilde{x})}{\partial(t, x)} \right| = 1$  leads to

$$(\mathcal{F}_{(x \rightarrow \xi)} F(t, \cdot; \omega))(\xi) = e^{-i(\omega \cdot \xi)t} \int_{\mathbb{R}^n} \int_{\mathbb{R}} e^{-i\tilde{x} \cdot \xi} e^{-i(-\omega \cdot \xi)\tilde{t}} \left( A_0 + \sum_{j=1}^n \omega_j A_j \right) (\tilde{t}, \tilde{x}) \, d\tilde{t} \, d\tilde{x},$$

where the right hand side of the above equation is the Fourier transform (in all variables) of  $A_0 + \sum_{j=1}^n \omega_j A_j$  at the point  $(-\omega \cdot \xi, \xi)$ . This equation can be rewritten as

$$e^{it\omega \cdot \xi} (\mathcal{F}_{(x \rightarrow \xi)} F(t, \cdot; \omega))(\xi) = \left( A_0 + \sum_{j=1}^n \omega_j A_j \right)^\wedge (-\omega \cdot \xi, \xi)$$

and we realize that since the right hand side is independent of  $t$ , so must be the left hand side. In particular when  $t = 0$  we have

$$\left( A_0 + \sum_{j=1}^n \omega_j A_j \right)^\wedge (-\omega \cdot \xi, \xi) = (\mathcal{F}_{(x \rightarrow \xi)} F(0, \cdot; \omega))(\xi) =: G(\xi; \omega). \quad (61)$$

Since the potentials  $A_j$  are smooth and compactly supported,  $F(0, \cdot; \cdot) : \mathbb{R}_x^n \times S^{n-1} \rightarrow \mathbb{R}$  is also smooth and compactly supported<sup>1</sup>, moreover (60) shows that it is uniformly bounded by  $C(n, \Omega) \|\Lambda_1 - \Lambda_2\|$ , hence

$$\begin{aligned} |G(\xi; \omega)| &= \left| \int_{\mathbb{R}^n} e^{-ix \cdot \xi} F(0, x; \omega) \, dx \right| \\ &\leq \|F(0, \cdot; \cdot)\|_{L^\infty(\mathbb{R}_x^n \times S^{n-1})} \text{Vol}(B_n(R)) \\ &\leq C(n, \Omega) R^n \|\Lambda_1 - \Lambda_2\|, \end{aligned} \quad (62)$$

which tells us that  $G$  is uniformly bounded in  $\mathbb{R}_\xi^n \times S^{n-1}$ .

We now turn our attention to the Fourier transform of the potentials. We want to obtain an estimate for  $|\widehat{A_j}(\tau, \xi)|$  on a conic set whose complement

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<sup>1</sup>This is because for  $|x|$  big enough the light rays with direction  $(1, \omega)$  emanating from the point  $(0, x)$  do not intersect the support of the potentials  $A_j$ .

contains the light cone  $\{(\tau, \xi) : |\tau| < |\xi|\}$  and use this estimate as well as an analytic continuation argument to obtain bounds for  $|\widehat{A}_j(\tau, \xi)|$  in the full space  $\mathbb{R}_\tau \times \mathbb{R}_\xi^n$ .

For  $(\tau, \xi)$  fixed with  $|\tau| < \frac{1}{2}|\xi|$  we know by considerations made in the previous section that we can find unit vectors  $\omega = \omega(\tau, \xi)$  parametrized by  $rS^{n-2}$  (an  $(n-2)$ -dimensional sphere with radius  $r$ ,  $\frac{\sqrt{3}}{2} \leq r \leq 1$ ), such that  $\tau + \omega(\tau, \xi) \cdot \xi = 0$  and satisfying  $\omega(\theta\tau, \theta\xi) = \omega(\tau, \xi)$  for any  $\theta > 0$ , i.e.,  $\omega(\tau, \xi)$  is homogenous of degree 0 in  $(\tau, \xi)$ .

We consider a maximal one dimensional sphere with radius  $r$  contained in  $rS^{n-2}$  and choose unit vectors  $\omega^{(1)}(\tau, \xi), \dots, \omega^{(n)}(\tau, \xi)$  forming the vertices of a regular polygon with  $n$  sides. We then consider the following set of  $n+1$  equations

$$\begin{cases} \widehat{A}_0(\tau, \xi) + \sum_{j=1}^n \omega_j^{(k)}(\tau, \xi) \widehat{A}_j(\tau, \xi) = G(\xi; \omega^{(k)}(\tau, \xi)) & k = 1, \dots, n \\ \frac{1}{\sqrt{\tau^2 + |\xi|^2}} \left( \tau \widehat{A}_0(\tau, \xi) + \sum_{j=1}^n \xi_j \widehat{A}_j(\tau, \xi) \right) = 0, \end{cases} \quad (63)$$

where the last equation is a simple consequence of the divergence condition  $\partial_t A_0(t, x) + \sum_{j=1}^n \partial_{x_j} A_j(t, x) = 0$ . Our goal is to show that this system is uniquely solvable for  $(\widehat{A}_0, \widehat{A}_1, \dots, \widehat{A}_n)$ .

In order to prove this statement it suffices to show that the matrix

$$M(\tau, \xi) = \begin{pmatrix} 1 & \omega_1^{(1)}(\tau, \xi) & \dots & \omega_n^{(1)}(\tau, \xi) \\ 1 & \omega_1^{(2)}(\tau, \xi) & \dots & \omega_n^{(2)}(\tau, \xi) \\ \dots & \dots & \dots & \dots \\ 1 & \omega_1^{(n)}(\tau, \xi) & \dots & \omega_n^{(n)}(\tau, \xi) \\ \frac{\tau}{\sqrt{\tau^2 + |\xi|^2}} & \frac{\xi_1}{\sqrt{\tau^2 + |\xi|^2}} & \dots & \frac{\xi_n}{\sqrt{\tau^2 + |\xi|^2}} \end{pmatrix}$$

is invertible. Notice that the entries of  $M(\tau, \xi)$  are homogeneous of degree 0 in  $(\tau, \xi)$  and the inverse, if it exists, will also have entries that are homogeneous of degree 0 in  $(\tau, \xi)$ .

To see that  $M(\tau, \xi)$  is indeed invertible we show the homogeneous system

$$\begin{cases} \widehat{A}_0(\tau, \xi) + \sum_{j=1}^n \omega_j^{(k)}(\tau, \xi) \widehat{A}_j(\tau, \xi) = 0 & k = 1, \dots, n \\ \frac{1}{\sqrt{\tau^2 + |\xi|^2}} \left( \tau \widehat{A}_0(\tau, \xi) + \sum_{j=1}^n \xi_j \widehat{A}_j(\tau, \xi) \right) = 0, \end{cases} \quad (64)$$

has no non-trivial solution. Once again, the considerations made in the previous section guarantee that the only potentials  $\mathcal{A} = (A_0, A_1, \dots, A_n)$  satisfying the first  $n$  equations are those of the form  $\widehat{A}_0(\tau, \xi) = \Phi(\tau, \xi)\tau$ ,  $\widehat{A}_j(\tau, \xi) = \Phi(\tau, \xi)\xi_j$ ,  $j = 1, \dots, n$ , for some smooth function  $\Phi$ . The last equation in the above system leads to  $\Phi(\tau, \xi)\sqrt{\tau^2 + |\xi|^2} = 0$  which in turn gives  $\Phi \equiv 0$  and  $\widehat{\mathcal{A}} = 0$ .

Since  $M(\tau, \xi)$  is invertible we can write

$$\widehat{A}_j(\tau, \xi) = \sum_{k=1}^n c_{k,j}(\tau, \xi) G(\xi; \omega^{(k)}(\tau, \xi)), \quad 1 \leq k \leq n, \quad 0 \leq j \leq n,$$

for some  $c_{k,j}(\tau, \xi)$  homogeneous of degree 0 in  $(\tau, \xi)$ . This homogeneity property as well as the uniform boundedness of  $G$  allows us to compute an estimate for the Fourier transform of the potentials  $A_j$  in the ray  $\{(\alpha\tau, \alpha\xi) : \alpha \in \mathbb{R}\}$

$$\begin{aligned} |\widehat{A}_j(\alpha\tau, \alpha\xi)| &= \left| \sum_{k=1}^n c_{k,j}(\alpha\tau, \alpha\xi) G(\alpha\xi; \omega^{(k)}(\alpha\tau, \alpha\xi)) \right| \\ &\leq \sum_{k=1}^n |c_{k,j}(\tau, \xi)| |G(\alpha\xi; \omega^{(k)}(\alpha\tau, \alpha\xi))| \\ &\leq C(n, \Omega) ||\Lambda_1 - \Lambda_2|| \sum_{k=1}^n |c_{k,j}(\tau, \xi)|, \end{aligned} \quad (65)$$

where in the last line of the previous inequality we used (62).

At this point it is convenient to recall that our initial goal is to obtain a uniform bound for  $\widehat{A}_j(\tau, \xi)$  in the set  $\{(\tau, \xi) : |\tau| \leq \frac{|\xi|}{2}\}$ . In view of (65) it suffices to work on the compact set  $\{(\tau, \xi) : \tau^2 + |\xi|^2 = 1, |\tau| \leq \frac{|\xi|}{2}\}$ . To obtain such a bound it is necessary to study the entries  $c_{k,j}(\tau, \xi)$  of the inverse of the matrix  $M(\tau, \xi)$ . It is a well know result that such entries can

be described by

$$c_{k,j}(\tau, \xi) = \frac{1}{\det M(\tau, \xi)} C_{j,k}(\tau, \xi)$$

where  $C_{j,k}(\tau, \xi)$  is the  $(j, k)$ -cofactor of  $M(\tau, \xi)$ .

Since on the set  $\{(\tau, \xi) : \tau^2 + |\xi|^2 = 1, |\tau| \leq \frac{|\xi|}{2}\}$  all the entries of  $M(\tau, \xi)$  have absolute value less or equal to one, and since  $C_{j,k}(\tau, \xi)$  consists of sums of products of  $n$  such entries, we have

$$|c_{k,j}(\tau, \xi)| \leq \frac{|C_{j,k}(\tau, \xi)|}{|\det M(\tau, \xi)|} \leq \frac{n}{|\det M(\tau, \xi)|}.$$

The quantity  $|\det M(\tau, \xi)|$  can be interpreted as the  $(n+1)$ -dimensional volume generated by the set of vectors  $\{(1, \omega^{(1)}(\tau, \xi)), \dots, (1, \omega^{(n)}(\tau, \xi)), (\tau, \xi)\}$ , however, since independent of the values  $(\tau, \xi)$  the vectors  $\omega^{(1)}(\tau, \xi), \dots, \omega^{(n)}(\tau, \xi)$  are chosen to be the vertices of a regular polygon, the  $n$ -dimensional volume  $V$  generated by  $\{(1, \omega^{(1)}(\tau, \xi)), \dots, (1, \omega^{(n)}(\tau, \xi))\}$ , is constant. We then have  $|\det M(\tau, \xi)| = V \times P(\tau, \xi)$  where  $P(\tau, \xi)$  is the projection of  $(\tau, \xi)$  into the linear subspace generated by  $\{(1, \omega^{(1)}(\tau, \xi)), \dots, (1, \omega^{(n)}(\tau, \xi))\}$ . This projection is given by  $C \sin \varphi$  where  $\varphi$  is the angle between  $(\tau, \xi)$  and said subspace. Since the vectors  $(1, \omega^{(k)}(\tau, \xi))$ ,  $k = 1, \dots, n$ , are located in the boundary of the light cone (i.e., the set  $\{(\tau, \xi) : |\tau| = |\xi|\}$ ), this angle is bounded below by  $\frac{\pi}{8}$ . Therefore on the set  $\{(\tau, \xi) : \tau^2 + |\xi|^2 = 1, |\tau| \leq \frac{|\xi|}{2}\}$  the value  $|\det M(\tau, \xi)|$  is uniformly bounded from below by  $V \sin \frac{\pi}{8}$  and

$$|c_{k,j}(\tau, \xi)| \leq \frac{n}{V \sin \frac{\pi}{8}}.$$

These observations combined with (65) give the uniform estimate

$$|\widehat{A}_j(\tau, \xi)| \leq C(n, \Omega) ||\Lambda_1 - \Lambda_2|| \quad (66)$$

on the set  $\{(\tau, \xi) : |\tau| \leq \frac{|\xi|}{2}\}$ .

Our next step is to obtain an upper bound for  $\widehat{A}_j(\tau, \xi)$ ,  $j = 0, \dots, n$ , on the complement of  $\{(\tau, \xi) : |\tau| \leq \frac{|\xi|}{2}\}$ . To obtain such estimate we first fix  $\tau$  and compute upper bounds for all lines that pass through the origin and are contained in the hyperplane  $\tau = \tau_0$ .

We will consider the case where the line corresponds to the  $\xi_n$ -axis, but before doing so, we will need some auxiliary results.

**Lemma 7.1.** *Consider the strip*

$$S = \{z = z_1 + iz_2 : z_1 \in \mathbb{R}, |z_2| < 2|\tau_0|\pi, \tau_0 \neq 0\}$$

*and the rays*

$$p_1 = \{z : -\infty < z_1 \leq -2|\tau_0|\pi, z_2 = 0\}, \quad p_2 = \{z : 2|\tau_0|\pi \leq z_1 < \infty, z_2 = 0\}$$

*in the complex plane  $\mathbb{C}$ .*

*If  $E = p_1 \cup p_2$  and  $G = S \setminus E$  is the strip with cuts along the rays  $p_1$  and  $p_2$ , we have*

$$\frac{2}{3} < \varpi(z, E, G) \leq 1, \quad (67)$$

*where  $\varpi(z, G, E)$  is the harmonic measure of  $E$  with respect to  $G$ .*

This statement is a very well known result about harmonic measures, its proof is mostly taken from [2] and it is included here for the purpose of self contention.

*Proof.* For  $h > 0$ , the map

$$\zeta(z) = \left( \frac{\exp(z\pi/h) - \exp(a\pi/h)}{\exp(z\pi/h) - \exp(-a\pi/h)} \right)^{\frac{1}{2}}$$

comformally transforms  $\overline{G}$  into the upper half plane  $\mathbf{H}^+ = \{\zeta = \zeta_1 + i\zeta_2 : \zeta_1 \in \mathbb{R}, \zeta_2 \geq 0\}$ . Under this mapping, the interval  $I = \{z : |z_1| < a, z_2 = 0\}$  transforms into the imaginary half axis  $\{\zeta : \zeta_1 = 0, \zeta_2 > 0\}$ . The boundary of  $G$  goes into the real axis and the set  $E = p_1 \cup p_2$  transforms into the subset of the real axis

$$E_1 = \{\zeta : \zeta_1 \leq -e^{a\pi/h}, \zeta_2 = 0\} \cup \{\zeta : |\zeta_1| \leq 1, \zeta_2 = 0\} \cup \{\zeta : \zeta_1 \geq e^{a\pi/h}, \zeta_2 = 0\}.$$

Then by the harmonic principle (see [4]), the values of the harmonic measures on  $E$ ,  $E_1$  with respect to the sets  $G$ ,  $\mathbf{H}^+$  agree, this is

$$\varpi(z, E, G) = \varpi(\zeta(z), E_1, \mathbf{H}^+).$$

We also know that the harmonic measure on the right hand side of the previous equation can be constructed by means of the Poisson integral for the upper half plane

$$\varpi(\zeta) = \frac{1}{\pi} \int_{-\infty}^{\infty} \chi_{E_1}(t) \frac{\zeta_2}{(t - \zeta_1)^2 + \zeta_2^2} dt \quad (68)$$



where  $\chi_{E_1(t)}$  is the characteristic function of  $E_1$ .

Since we are interested in the image of  $I$  under the map  $\zeta(z)$  and since this image is precisely the positive imaginary axis, we may assume without loss of generality that  $\zeta = i\zeta_2$ ,  $\zeta_2 > 0$ . From (68) we obtain

$$\varpi(\zeta(z), E_1, \mathbf{H}^+) = \frac{2}{\pi} \left( \frac{\pi}{2} - \arctan \frac{\zeta_2(\exp(a\pi/h) - 1)}{(\zeta_2)^2 + \exp(a\pi/h)} \right) \quad (69)$$

Choosing  $h = a\pi$  and using the inequality

$$\arctan \frac{\zeta_2(e-1)}{(\zeta_2)^2 + e} \leq \arctan \frac{(e-1)}{2e^{\frac{1}{2}}} \quad (70)$$

we obtain

$$\frac{2}{\pi} \left( \frac{\pi}{2} - \arctan \frac{(e-1)}{2e^{\frac{1}{2}}} \right) \leq \varpi \leq 1$$

and we conclude that

$$\frac{2}{3} \leq \varpi(z, E, G) \leq 1.$$

□

Based on this result we want to ‘embed’ the  $\xi_n$ -axis into said strip and use the bounds on the harmonic measure. To do so we realize that since the potentials  $A_j$ ,  $j = 0, \dots, n$ , are compactly supported, the functions  $\widehat{A}_j(\tau_0, \xi)$  admits an analytic extension in  $\xi_n$  into the complex plane. If we let

$$\begin{aligned} \Pi &= \{\nu = (\nu_1, \nu_2) : \nu_1 \in \mathbb{R}, |\nu_2| < 2|\tau_0|\pi, \tau_0 \neq 0\}, \\ q_1 &= \{\nu = (\nu_1, \nu_2) : -\infty < \nu_1 \leq -2|\tau_0|, \nu_2 = 0\}, \\ q_2 &= \{\nu = (\nu_1, \nu_2) : 2|\tau_0| \leq \nu_1 < \infty, \nu_2 = 0\} \end{aligned}$$

and restrict ourselves to the  $\xi_n$ -axis (i.e.,  $\xi_1 = \dots = \xi_{n-1} = 0$ ), the estimate (67) leads to

$$\frac{2}{3} < \varpi(\nu, E_1, G_1) \leq 1,$$

where  $E_1 = q_1 \cup q_2$  and  $G_1 = \Pi \setminus E_1$ .

Denoting by  $v_j(\nu) = \widehat{A}_j(2\tau_0, 0, \dots, 0, \nu)$ , the restriction of  $\widehat{A}_j$  to the  $\xi_n$ -axis, we have by the two-constant theorem (see [16] Theorem 9.4.5)

$$|v_j(\nu)| \leq m_j^{\frac{2}{3}} M_j^{\frac{1}{3}} \quad (71)$$

where  $m_j$  and  $M_j$  are the respective upper bounds of the modulus of  $v(\nu)$  on the rays  $q_1$  and  $q_2$  and on the lines  $\{(\nu_1, \nu_2) : \nu_1 \in \mathbb{R}, \nu_2 = -2|\tau_0|\pi\}$  and  $\{(\nu_1, \nu_2) : \nu_1 \in \mathbb{R}, \nu_2 = 2|\tau_0|\pi\}$ .

At this point it is worth to point out that the rays  $q_1$  and  $q_2$  are contained in the set  $\{(\tau, \xi) : |\tau| \leq \frac{|\xi|}{2}\}$  and that we have already computed an estimate for  $|v_j(\nu)|$  in that region (equation 66). To compute  $M_j$  we resort to the identities

$$v_j(\nu) = C(\pi, n) \int_{\mathbb{R}} e^{-i(\nu_1 + i\nu_2)x_n} W_j(2\tau_0, 0, \dots, 0, x_n) dx_n$$

with  $W_j$  the Fourier transform of  $A_j$  in all variables except  $x_n$ . Next, we realize that these functions are compactly supported in  $x_n$  and the above integrand is nonzero only on a finite subset of the real numbers. Hence

$$|v_j(\nu)| \leq \sup_{x_n \in (-a(\Omega), a(\Omega))} |W_j(2\tau_0, 0, \dots, 0, x_n)| \int_{-a(\Omega)}^{a(\Omega)} e^{2|\tau_0|\pi x_n} dx_n,$$

where  $a(\Omega)$  is a positive number bigger than  $\text{diam}(\Omega)$ . Since  $\nu = \nu_1 + i\nu_2$  is restricted to the strip  $\Pi$  we have

$$|v_j(\nu)| \leq C(\Omega, n) \frac{e^{2|\tau_0|\pi}}{2|\tau_0|\pi},$$

and in particular when  $\nu$  is a real number satisfying  $-2|\tau_0| < \nu < 2|\tau_0|$  we have by (71)

$$|v_j(\nu)| \leq C(\Omega, n) \frac{e^{\frac{2|\tau_0|\pi}{3}} m_j^{\frac{2}{3}}}{2|\tau_0|^{\frac{1}{3}}}.$$

All this arguments can be carried out to the case where a line is contained in the hyperplane  $\tau = \tau_0$ , passes through the origin but is not parallel to any of the axes. Finally using (66) we obtain an estimate for the Fourier transform of the potentials in the set  $\{(\tau, \xi) : |\tau| > \frac{|\xi|}{2}\}$ , namely

$$|\widehat{A_j}(\tau, \xi)| \leq C(\Omega, n) \frac{e^{\frac{2|\tau|\pi}{3}} |||\Lambda_1 - \Lambda_2|||^{\frac{2}{3}}}{|\tau|^{\frac{1}{3}}}. \quad (72)$$

From estimates (66) and (72) we can establish the desired estimate for the vector potentials. The general idea is to use the inequality  $\|f\|_{L^\infty} \leq \|\widehat{f}\|_{L^1}$  and partition  $\mathbb{R}_\tau \times \mathbb{R}_\xi^n$  in an appropriate way.

From the Fourier inversion formula we have

$$A_j(t, x) = C(\pi) \iint_{\mathbb{R}_\tau \times \mathbb{R}_\xi^n} e^{i(t\tau + x \cdot \xi)} \widehat{A_j}(\tau, \xi) d\tau d\xi \quad (73)$$

and by taking absolute values we have for any  $\rho_1 > 0$

$$\begin{aligned} |A_j(t, x)| &\leq C(\pi) \iint_{\mathbb{R}_\tau \times \mathbb{R}_\xi^n} |\widehat{A_j}(\tau, \xi)| d\tau d\xi \\ &\leq C(\pi) \iint_{B(\rho_1)} |\widehat{A_j}(\tau, \xi)| d\tau d\xi \\ &\quad + C(\pi) \iint_{B(\rho_1)^c} |\widehat{A_j}(\tau, \xi)| d\tau d\xi \\ &= I_1 + I_2, \end{aligned}$$

where  $B(\rho_1)$  denotes the  $(n+1)$ -dimensional ball  $B(\rho_1) = \{(\tau, \xi) : |\tau|^2 + |\xi|^2 \leq \rho_1^2\}$ .

To obtain a bound for  $I_2$  we recall that the potentials  $A_j$ ,  $j = 1, \dots, n$ , are  $C^\infty$  in  $t$  and  $x$ . Hence for any  $\beta > 0$  and any  $\rho_1 > 0$  if  $|\tau|^2 + |\xi|^2 \leq \rho_1^2$

$$|\widehat{A_j}(\tau, \xi)| \leq \frac{C}{(|\tau|^2 + |\xi|^2)^{\frac{\beta}{2}}}.$$

If  $\beta > n + 2$  the integral  $I_2$  converges. Moreover, the estimate

$$I_2 = \iint_{B(\rho_1)^c} |\widehat{A_j}(\tau, \xi)| d\tau d\xi \leq \frac{C(n)}{\rho_1^{\beta-n-1}} \leq \frac{C(n)}{\rho_1} \quad (74)$$

holds.

To estimate  $I_1$  we break up the ball  $B(\rho_1)$  into two smaller pieces

$$\mathcal{C}_1 = B(\rho_1) \cap \left\{(\tau, \xi) : |\tau| < \frac{\rho_1}{\sqrt{5}}\right\} \quad \text{and} \quad \mathcal{C}_2 = B(\rho_1) \cap \left\{(\tau, \xi) : |\tau| \geq \frac{\rho_1}{\sqrt{5}}\right\}.$$

Then

$$I_1 \leq \iint_{\mathcal{C}_1} |\widehat{A_j}(\tau, \xi)| d\tau d\xi + \iint_{\mathcal{C}_2} |\widehat{A_j}(\tau, \xi)| d\tau d\xi,$$

and since  $\mathcal{C}_1$  is a compact subset of  $B(\rho_1)$  we have

$$I_1 \leq C\rho^{n+1} + \iint_{\mathcal{C}_2} |\widehat{A_j}(\tau, \xi)| d\tau d\xi.$$

The advantage of this decomposition is that  $\mathcal{C}_2$  is contained in the set  $\{(\tau, \xi) : |\tau| > \frac{|\xi|}{2}\}$  and that on this set  $|\tau|$  is bounded below by  $\frac{\rho_1}{\sqrt{5}}$ . Thus by (72)

$$I_2 \leq C\rho^{n+1} + \frac{C(\Omega, n) e^{\frac{2\rho\pi}{3}} |||\Lambda_1 - \Lambda_2|||^\frac{2}{3} \rho^{n+1}}{\rho^\frac{1}{3}} \quad (75)$$

where we have set  $\rho = \frac{\rho_1}{\sqrt{5}}$ .

Equations (73)-(75) lead to

$$\begin{aligned} |A_j(t, x)| &\leq C(\Omega, n) \left[ \frac{1}{\rho} + \rho^{n+1} + \rho^{n+\frac{2}{3}} e^{\frac{2\rho\pi}{3}} |||\Lambda_1 - \Lambda_2|||^\frac{2}{3} \right] \\ &\leq C(\Omega, n) \left[ \frac{1}{\rho} + \rho^{n+\frac{2}{3}} e^{\frac{2\rho\pi}{3}} |||\Lambda_1 - \Lambda_2|||^\frac{2}{3} \right]. \end{aligned} \quad (76)$$

Now the idea is to choose  $\rho$  small enough so that the two terms in the the right hand side of (76) are comparable. In other words we want  $\rho$  to satisfy the identity

$$\frac{C}{\rho} = \rho^{n+\frac{2}{3}} e^{\frac{2\rho\pi}{3}} |||\Lambda_1 - \Lambda_2|||^\frac{2}{3}$$

for some constant  $C$ . By taking logarithms on both sides of the previous equation we obtain the equivalent identity

$$2 \log \frac{C}{|||\Lambda_1 - \Lambda_2|||} = (3n + 5) \log \rho + 2\pi\rho, \quad (77)$$

and since the right hand side of (77) is one to one when  $\rho > 0$  we know that it admits a unique solution.

On the other hand, the inequality  $\log \rho \leq \rho$  for positive  $\rho$  as well as (77) lead to

$$2 \log \frac{C}{|||\Lambda_1 - \Lambda_2|||} \leq (3n + 5 + 2\pi)\rho,$$

or

$$\frac{1}{\rho} \leq \frac{3n + 5 + 2\pi}{2} \left[ \log \frac{C}{|||\Lambda_1 - \Lambda_2|||} \right]^{-1}$$

and equation (76) becomes

$$|A_j(t, x)| \leq C(\Omega, n) \left[ \log \frac{C}{|||\Lambda_1 - \Lambda_2|||} \right]^{-1}.$$

Summarizing, we have proved the following

**Theorem 7.1.** *Suppose that the vector potentials  $\mathcal{A}^{(l)} = (A_0^{(l)}, \dots, A_n^{(l)})$ ,  $l = 1, 2$ , are real valued, compactly supported and  $C^\infty$  in  $t$  and  $x$ . Let  $\mathcal{A} = (A_0, A_1, \dots, A_n)$  where  $A_j = A_j^{(1)} - A_j^{(2)}$  and suppose that the divergence condition*

$$\operatorname{div} \mathcal{A} = \partial_t A_0(t, x) + \sum_{j=1}^n \partial_{x_j} A_j(t, x) = 0$$

*holds. If  $\Lambda_l$  represents the Dirichlet to Neumann operator associated to the hyperbolic problem (1)-(4), then the stability estimate*

$$\max_{0 \leq j \leq n} \left\| A_j^{(1)}(t, x) - A_j^{(2)}(t, x) \right\|_{L^\infty(\mathbb{R}_t \times \mathbb{R}_x^n)} \leq C(\Omega, n) \left[ \log \frac{C}{|||\Lambda_1 - \Lambda_2|||} \right]^{-1} \quad (78)$$

*holds for  $\Lambda_1, \Lambda_2$  satisfying  $|||\Lambda_1 - \Lambda_2||| \ll 1$ .*

*Proof.* The hypothesis of the theorem guarantee that conditions *i)* and *ii)* hold and the only condition that is not automatically satisfied from the hypothesis is condition *iii)*. However a simple rescaling of the vector potentials  $A_j \rightarrow A'_j = \frac{1}{\alpha} A_j$ , with  $\alpha$  the supremum of all ray integrals of the potentials as well as a similar rescaling of the coordinate axis show that the estimate (78) holds for the potentials  $A'_j$ . In turn, this implies a similar estimate for the original potentials  $A_j$ .  $\square$

## 8 Presence of obstacles

One variation of the above problem consists in the introduction of convex obstacles inside the domain  $\Omega$ . That is, let  $\Omega_k$ ,  $1 \leq k \leq M$ , be simply connected bounded domains in  $\mathbb{R}^n$ ,  $n \geq 3$  and let  $D = \Omega \setminus \cup_{k=1}^M \Omega_k$ . If we consider again the equation

$$(-i\partial_t + A_0(t, x))^2 u - \sum_{j=1}^n (-i\partial_{x_j} + A_j(t, x))^2 u + V(t, x)u = 0 \quad \text{in } \mathbb{R} \times \Omega,$$

$$\begin{aligned} u(t, x) &= 0 & \text{for } t \ll 0 \\ u(t, x) &= f(t, x) & \text{on } \mathbb{R} \times \partial\Omega, \end{aligned}$$

with the additional condition

$$u(t, x)|_{\mathbb{R} \times \partial\Omega_k} = 0, \quad 1 \leq k \leq M. \quad (79)$$

Then we can prove as in the case of no obstacles that the vector valued potential  $\mathcal{A}(t, x)$  satisfies

$$\mathcal{P}\mathcal{A}(t, x; \omega) = \int_{-\infty}^{\infty} \sum_{j=0}^n \omega_j A_j(t + s, x + s\omega) ds = 0 \quad (80)$$

for any ray

$$\gamma = \{(t, x) + s(1, \omega) \mid s \in \mathbb{R}\} \quad (81)$$

not intersecting the obstacles  $\mathbb{R} \times \Omega_k$ ,  $1 \leq k \leq M$ . We will show that under some conditions the vector and scalar potentials can be recovered outside these obstacles.

As a warm up let us consider the case where  $n = 3$ , there is only one obstacle and the integrals over light rays are zero for a smooth scalar function  $f = f(t, x)$  satisfying the support condition  $f(t, x) = 0$  when  $|x| > R$ , and the growth condition  $|f(t, x)| \leq C(1 + |t|)^N$  for some integer  $N$ .

Under these settings, for  $\tilde{x}$  an arbitrary point in  $D$ , we can find a two dimensional plane  $P$  in  $\mathbb{R}^3$  containing  $\tilde{x}$  such that  $P$  misses  $\Omega_1$ , this is  $P \cap \Omega_1 = \emptyset$ . We then realize that the set of rays  $\gamma$  of the form (81) that are contained in the three dimensional space  $\mathbb{R}_t^1 \times P$  and pass through  $\tilde{x}$ , can be parametrized by the one dimensional sphere  $S^1$ . This in turn, allows us to use our previous considerations for a scalar function in two dimensions (c.f. *Ramm–J. Sjöstrand*, [23]) to conclude that  $f$  vanishes in  $\mathbb{R}_t^1 \times P$ . Since  $P$  could be any two dimensional plane not intersecting  $\Omega_1$  and  $\tilde{x}$  was selected to be an arbitrary point in  $D = \Omega \setminus \Omega_1$  we conclude that  $f = 0$  on  $D \times \mathbb{R}_t^1$ .

Unfortunately for the vector potential things are going to be slightly harder and we will have to make some geometric considerations in order to obtain an equivalent result.

We shall impose the following restriction on the problem:

(G1) The obstacles are convex and

- when  $n \geq 4$ , for each  $\tilde{x} \in D$  there exists a two dimensional plane  $P$  passing through  $\tilde{x}$  such that  $P$  does not intersect any of the obstacles.
- when  $n = 3$  there is only one obstacle.

**Theorem 8.1.** *If condition (G1) holds and the Dirichlet to Neumann operators are equal ( $\Lambda_1 = \Lambda_2$ ), then the potentials  $\mathcal{A}^{(1)}$  and  $\mathcal{A}^{(2)}$  are gauge equivalent, i.e., there exists a smooth function  $\varphi$  such that  $\mathcal{A}^{(1)} - \mathcal{A}^{(2)} = d\varphi$ .*

*Proof.* Let us proceed first by assuming that  $n = 3$ .

Regarding the vector potential  $\mathcal{A}(t, x)$  as a 1-form, the goal will be to prove that the 2-form  $d\mathcal{A}$  vanishes outside  $\Omega_1$ .

For  $\tilde{x}$  not in  $\Omega_1$ , using condition (G1) we can find a 2-dimensional plane  $P \subset \mathbb{R}_x^3$  not intersecting the obstacle and we can choose three linearly independent unit vectors  $\eta_j$ ,  $j = 1, 2, 3$  close to  $P$  such that any plane  $P_{jp}$ ,  $1 \leq j, p \leq 3$  passing through  $\tilde{x} + \eta_j$  and  $\tilde{x} + \eta_p$  does not meet  $\Omega_1$ . Next, introducing coordinates  $x' = (x'_1, x'_2, x'_3)$  in  $\mathbb{R}_x^3$  by the formula  $x - \tilde{x} = x'_1\eta_1 + \dots + x'_3\eta_3$  and denoting by  $\mathcal{A}'$  and  $B'$  the 1-form and 2-form mentioned above expressed in the new coordinate system, we have

$$\mathcal{A}' = A'_0 dx'_0 + A'_1 dx'_1 + A'_2 dx'_2 + A'_3 dx'_3 \quad (82)$$

$$B'(t, x') = \sum_{0 \leq p < j \leq 3} b'_{jp}(t, x') dx'_j \wedge dx'_p \quad (83)$$

where  $x'_0 = t$  and

$$b'_{jp} = \frac{\partial A'_j}{\partial x'_p} - \frac{\partial A'_p}{\partial x'_j}. \quad (84)$$

The restriction of  $B'$  to the 3-dimensional space  $\mathbb{R}_t^1 \times P_{jp}$  is given by

$$B'|_{\mathbb{R}_t^1 \times P_{jp}} = b'_{0j} dx'_0 \wedge dx'_j + b'_{0p} dx'_0 \wedge dx'_p + b'_{jp} dx'_j \wedge dx'_p \quad (85)$$

and since there are no obstacles in  $\mathbb{R}_t^1 \times P_{jp}$  theorem (6.3) gives

$$A'|_{\mathbb{R}_t^1 \times P_{jp}} = \nabla \varphi_{jp}(t, x'_j, x'_p) \quad (86)$$

where  $\nabla$  is the gradient in the  $(t, x'_j, x'_p)$  and  $\varphi_{jp}$  is a smooth function compactly supported in  $x'_{jp} = (x'_j, x'_p)$ . Clearly then equation (84) gives  $B'|_{\Pi_{jp} \times \mathbb{R}_t^1} = 0$ .

As the above discussion holds for all three dimensional spaces parallel and close to  $P_{jp}$  we see that  $B'$  and thus  $B = d\mathcal{A}$  vanishes for  $x$  near  $\tilde{x}$  and for all values of  $t$ . Being that  $\tilde{x}$  is an arbitrary point not in the obstacle, we get that  $d\mathcal{A}$  vanishes outside  $\Omega_1$  and thus, since we are working in a simply connected domain, the vector potential is the gradient of a smooth function.

When there are two or more obstacles and  $\tilde{x}$  is a point not lying in any of them, we resort to the geometric condition (G1) to find a two dimensional plane  $P_0$  such that  $P_0$  intersects no obstacles and choose  $n$  linearly independent unit vectors  $\eta_j, \dots, \eta_n$  close to  $P_0$  in such a way that  $\eta_1, \eta_2 \in P_0$  and the planes  $P_{jp}$ ,  $1 \leq j, p \leq n$ , do not intersect any obstacle.

As before, we introduce new coordinates  $x' = (x'_1, \dots, x'_n)$  given by  $x - \tilde{x} = x'_1 \eta_1 + \dots + x'_n \eta_n$ , and express  $\mathcal{A}$  and  $B$  in terms of these new coordinates. If we now consider the restrictions of  $B$  to the spaces  $\mathbb{R}_t^1 \times P_{jp}$ ,  $1 \leq j, p \leq n$ , we realize that we are back into the previous case. Making use of the fact that  $\mathcal{A}'|_{\mathbb{R}_t^1 \times P_{jp}} = \nabla \varphi_{jp}(t, x'_j, x'_p)$  we obtain via equation (84) that

$$B'|_{\mathbb{R}_t^1 \times P_{jp}} = 0, \quad (87)$$

which as before leads to  $B = 0$  and hence  $\mathcal{A}(t, x) = \nabla_{t,x} \Psi(t, x)$  for some smooth  $\Psi$ .  $\square$

When condition (G1) fails to hold, we can still recover some information regarding the difference of the vector potentials provided that  $d\mathcal{A} = 0$ . As we did before, let us regard vector potentials as 1-forms and let us consider the case when we have 2 obstacles inside the domain  $\Omega$ .

This time it might be the case that the domain is not simply connected and that for some close path  $\gamma$  the integral  $\int_\gamma \mathcal{A} \cdot d\bar{x}$  is not zero (here we set  $\bar{x} = (t, x)$ ), however, the condition  $d\mathcal{A} = 0$  guarantees that it only depends on the homotopy class of  $\gamma$ .

With this consideration in mind we want to be able to “span” every possible homotopy class representative  $\gamma$  of the domain  $\Omega$  by using light rays. In other words, we would like to impose on our domain the geometric condition (G2): every homotopy representative can be continuously contour-deformed into light rays.

If this geometric condition is met, we can write for  $\gamma_1$  an arbitrary simple closed path in  $\Omega$  surrounding the first of the obstacles

$$\int_{\gamma_1} \mathcal{A}(\bar{x}) \cdot d\bar{x} = \int_{\ell_1} \mathcal{A}(\bar{x}) \cdot d\bar{x} + \int_{\ell_2} \mathcal{A}(\bar{x}) \cdot d\bar{x} + \dots + \int_{\ell_r} \mathcal{A}(\bar{x}) \cdot d\bar{x},$$

where the set of light rays  $\ell_1, \ell_2, \dots, \ell_r$ , surround the first of the obstacles and are such that they do not intersect the second obstacle. Then by the previous arguments regarding the construction of geometric optics solutions and Green’s formula we have if  $\mathcal{A}^{(1)}$  and  $\mathcal{A}^{(2)}$  correspond to equal Dirichlet to Neumann operators

$$\int_{\gamma_1} (\mathcal{A}^{(1)}(\bar{x}) - \mathcal{A}^{(2)}(\bar{x})) \cdot d\bar{x} = 2\pi m_1.$$



Proceeding in a similar fashion, we have for the second obstacle and a contour  $\gamma_2$  surrounding it

$$\int_{\gamma_2} (\mathcal{A}^{(1)}(\bar{x}) - \mathcal{A}^{(2)}(\bar{x})) \cdot d\bar{x} = 2\pi m_2.$$

Then, for an arbitrary closed contour  $\gamma$  we have (after a contour deformation argument) that

$$\int_{\gamma=c_1\gamma_1+c_2\gamma_2} (\mathcal{A}^{(1)}(\bar{x}) - \mathcal{A}^{(2)}(\bar{x})) \cdot d\bar{x} = 2\pi c_1 m_1 + 2\pi c_2 m_2.$$

where  $c_1$  and  $c_2$  are two integers. We now let  $\Theta(\bar{x})$ , be the function that computes the angle between the projection of  $\bar{x}$  into the hyperplane  $t = 0$  and the vector  $(0, 0, \dots, 0, 1)$ , and for  $j = 1, 2$  we set  $\Theta_j(\bar{x}) = \Theta(x - p_j)$ , where  $p_j$  is any point inside the obstacle  $j$ . Then the functions  $\Theta_j$ , compute the ‘angle that a vector makes inside the obstacle  $j$ ’ and we have

$$\int_{\gamma} (\mathcal{A}^{(1)}(\bar{x}) - \mathcal{A}^{(2)}(\bar{x}) - m_1\Theta_1(\bar{x}) - m_2\Theta_2(\bar{x})) \cdot d\bar{x} = 0$$

for any closed contour  $\gamma$ . Therefore

$$\mathcal{A}^{(1)}(\bar{x}) - \mathcal{A}^{(2)}(\bar{x}) - m_1\Theta_1(\bar{x}) - m_2\Theta_2(\bar{x}) = \partial_{\bar{x}}\varphi(\bar{x}) \quad (88)$$

for some function  $\varphi$ .

We can certainly say more, if  $\gamma(\bar{x}_0; \bar{x})$  is any curve joining the points  $\bar{x} = (t, x)$  and  $\bar{x}_0 = (t_0, x_0)$ , and we let

$$C_0(\bar{x}) = \exp \left( -i \int_{\gamma(\bar{x}_0; \bar{x})} (m_1\Theta_1(\bar{x}) - m_2\Theta_2(\bar{x})) \cdot d\bar{x} \right),$$

then

$$\frac{i}{C_0(\bar{x})} \partial_{\bar{x}} C_0(\bar{x}) = m_1\Theta_1(\bar{x}) - m_2\Theta_2(\bar{x})$$

and equation (88) can be reewritten as

$$\mathcal{A}^{(1)}(\bar{x}) - \mathcal{A}^{(2)}(\bar{x}) - \frac{i}{C_0(\bar{x})} \partial_{\bar{x}} C_0(\bar{x}) = \partial_{\bar{x}}\varphi(\bar{x})$$

or

$$\mathcal{A}^{(1)}(\bar{x}) - \mathcal{A}^{(2)}(\bar{x}) = \frac{i}{C(\bar{x})} \partial_{\bar{x}} C(\bar{x})$$

where  $C(\bar{x}) = C_0(\bar{x}) \exp(i\varphi(\bar{x}))$ .

To conclude this section let us consider the case when we have any number of obstacles,  $d\mathcal{A} = 0$ ,  $\mathcal{A} = \mathcal{A}^{(1)} - \mathcal{A}^{(2)}$ ,  $\Omega$  is multi-connected and condition (G2) holds.

**Theorem 8.2.** *If  $d\mathcal{A}^{(1)} - d\mathcal{A}^{(2)} = 0$ , condition (G2) is satisfied and the Dirichlet to Neumann operators are equal ( $\Lambda_1 = \Lambda_2$ ). Then the potentials  $\mathcal{A}^{(1)}$  and  $\mathcal{A}^{(2)}$  are gauge equivalent, i.e., there exists a smooth function  $C = C(\bar{x})$  such that  $|C| = 1$  for  $\bar{x} = (t, x) \in \mathbb{R} \times \partial\Omega$  and  $\mathcal{A}^{(1)} - \mathcal{A}^{(2)} = \frac{i}{C(\bar{x})} \partial_{\bar{x}} C(\bar{x})$ .*

*Proof.* Let  $G((-T, T) \times \Omega)$  denote the gauge group corresponding to the set  $(-T, T) \times \Omega$  and let  $\ell_1, \dots, \ell_p$  be a basis for the homology group, this is,  $\gamma = n_1 \ell_1 + \dots + n_p \ell_p$  for any closed contour  $\gamma$ . Since  $d\mathcal{A} = 0$  the integrals  $\int_{\gamma} \mathcal{A}(\bar{x}) \cdot d\bar{x}$  depend only on the homotopy class of  $\gamma$ . By condition (G2) the basis of the homology group can be spanned by light rays that do not intersect any of the obstacles, hence if  $\mathcal{A}^{(1)}$  and  $\mathcal{A}^{(2)}$  correspond to gauge equivalent Dirichlet-to-Neumann operators, we have as before that

$$e^{i \int_{\ell_k} \mathcal{A}^{(1)}(\bar{x}) \cdot d\bar{x}} = e^{i \int_{\ell_k} \mathcal{A}^{(2)}(\bar{x}) \cdot d\bar{x}}.$$

Therefore for any closed curve  $\gamma$

$$\int_{\gamma} (\mathcal{A}^{(1)}(\bar{x}) - \mathcal{A}^{(2)}(\bar{x})) \cdot d\bar{x} = 2\pi q, \quad q \in \mathbb{Z},$$

which in turn implies the existence of a function  $C(t, x) \in G((-T, T) \times \Omega)$  such that

$$\mathcal{A}^{(1)}(\bar{x}) - \mathcal{A}^{(2)}(\bar{x}) = \frac{i}{C(\bar{x})} \partial_{\bar{x}} C(\bar{x}).$$

□

## Appendix A: Auxiliary results

### Orthogonal complement of a set in $\mathbb{R}^m$

**Lemma:** *The orthogonal complement of the set*

$$E = \{(1, \omega) : \omega \in S^{(n-1)}, \tau + \omega \cdot \xi = 0, |\tau| < |\xi|\}$$

*is a one dimensional subspace of  $\mathbb{R}^{n+1}$ .*

Let us start off with some linear algebra facts.

Let  $m \geq 2$ . If  $A \subseteq B \subseteq \mathbb{R}^m$  with

$$A \subseteq B \subseteq \text{Span}(A) = \left\{ \sum_{p=1}^r \alpha_p a_p : \alpha_p \in \mathbb{R}, a_p \in A, r \in \mathbb{N} \right\},$$

then

$$A^\perp = B^\perp = \text{Span}(A)^\perp.$$

Indeed, since orthogonal complements reverse inclusions we have  $\text{Span}(A)^\perp \subseteq B^\perp \subseteq A^\perp$ . Also, if  $x \cdot a = 0$  for all  $a \in A$ , then  $x \cdot \sum_{p=0}^r \alpha_p a_p = \sum_{p=0}^r \alpha_p (x \cdot a_p) = 0$ , which shows that  $A^\perp \subseteq \text{Span}(A)^\perp$ . Therefore  $A^\perp = B^\perp = \text{Span}(A)^\perp$ .

Denoting by  $\text{CH}(A)$  the *convex hull* of  $A$  and by  $\mathcal{C}(A)$  the *cone spanned* by  $A$  we have

$$\begin{aligned} \text{CH}(A) &= \left\{ \sum_{p=1}^r \alpha_p a_p : \sum_{p=1}^r \alpha_p = 1, 0 \leq \alpha_p \leq 1, a_p \in A, r \in \mathbb{N} \right\} \\ \mathcal{C}(A) &= \{ta : t \in \mathbb{R}^+, a \in A\}. \end{aligned}$$

Clearly  $A \subseteq \text{CH}(A) \subseteq \mathcal{C}(\text{CH}(A))$  and since both sets contain particular linear combinations of elements of  $A$  we also have  $\text{Span}(\mathcal{C}(\text{CH}(A))) = \text{Span}(A)$ , hence,

$$\text{Span}(\mathcal{C}(\text{CH}(A)))^\perp = \text{Span}(A)^\perp.$$

We want to apply these remarks to the set  $E$  but before doing so let us recall that for  $|\tau| < |\xi|$  the vectors  $\omega$  satisfying  $|\omega| = 1$ ,  $\tau + \omega \cdot \xi = 0$ , can be parametrized by  $S^{n-2}$ . Since rotations are non-singular transformations we can compute instead the dimension of the orthogonal complement of the set

$$\tilde{E} = \{(1, \omega_1, \dots, \omega_{n-1}, a) : \omega_1^2 + \dots + \omega_{n-1}^2 = 1 - a^2\},$$

where  $0 \leq a < 1$  is a fixed number. Taking into account our previous observations we then have

$$\begin{aligned}
\tilde{E}^\perp &= \text{Span}\left(\mathcal{C}(\text{CH}(\tilde{E}))\right)^\perp \\
&= \text{Span}\left(\mathcal{C}\left(\{(1, \omega_1, \dots, \omega_{n-1}, a) : \omega_1^2 + \dots + \omega_{n-1}^2 \leq 1 - a^2\}\right)\right)^\perp, \\
&= \text{Span}\left(\{(t, t\theta_1, \dots, t\theta_{n-1}, ta) : t, \theta_1, \dots, \theta_{n-1} \in \mathbb{R}, \theta_1^2 + \dots + \theta_{n-1}^2 \leq 1 - a^2\}\right)^\perp
\end{aligned}$$

and we can see that  $\tilde{E}^\perp$  is a one dimensional subspace of  $\mathbb{R}^{n+1}$  since clearly  $\text{Span}\left(\mathcal{C}(\text{CH}(\tilde{E}))\right)$  is  $n$ -dimensional.

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